1. Let
$$f(x) = \frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{3}{2}x^2 + 1.$$

(a) Find the critical numbers of f.

(b) Find the intervals of increase and decrease of f.

(a) $f'(x) = x^3 - 2x^2 - 3x = 0$

$x(x^2 - 2x - 3) = 0$

$x(x-3)(x+1) = 0$

$x = 0, 3, -1$

(b)

Interval	x	$x-3$	$x+1$	$f'(x)$	
$(-\infty, -1)$	$-$	$-$	$-$	$-$	dec.
$(-1, 0)$	$-$	$-$	$+$	$+$	increas.
$(0, 3)$	$+$	$-$	$+$	$-$	dec.
$(3, \infty)$	$+$	$+$	$+$	$+$	increas.

2. Calculate the limit, if it exists.

$$\lim_{x\to 0} \frac{\tan 4x}{x}.$$

$$\lim_{x\to 0} \frac{\tan 4x}{x} = \lim_{x\to 0} \frac{1}{x} \frac{\sin 4x}{\cos 4x}$$

$$= \lim_{x\to 0} \frac{4}{\cos 4x} \frac{\sin 4x}{4x}$$

$$= \lim_{x\to 0} \frac{4}{\cos 4x} \lim_{4x\to 0} \frac{\sin 4x}{4x} = \left(\frac{4}{1}\right)(1) = 4$$

3. Calculate

(a) the slope of the tangent to the curve

$$x \sin y = x + 1$$

at $(-1, 0)$.

(b) the derivative of

$$f(x) = \int_{-1}^{x^2} \sqrt{1+t^4}\, dt$$

at $x = 2$.

(a) $\sin y + x \cos y \dfrac{dy}{dx} = 1$

$\sin 0 + (-1) \cos 0 \dfrac{dy}{dx} = 1$

$-\dfrac{dy}{dx} = 1 \qquad \dfrac{dy}{dx} = -1$

(b) $g(x) = \int_{-1}^{x} \sqrt{1+t^4}\,dt \qquad g'(x) = \sqrt{1+x^4}$

$f(x) = g(x^2) \qquad f'(x) = g'(x^2)(2x)$

$= \sqrt{1+(x^2)^4}\,(2x)$

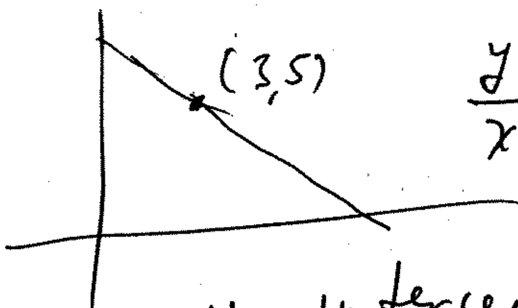
$f'(2) = \sqrt{1+(4)^4}\,(4)$

4. A line of negative slope through $(3, 5)$ bounds a triangular region of the first quadrant of the plane.

(a) Write the area of the triangle as a function of the slope of the line.

(b) Find the slope that makes the area of the triangular region as small as possible.

(a)



$$\frac{y-5}{x-3} = m$$

y-intercept: $y - 5 = -3m \quad y = -3m + 5$

x-intercept: $m(x-3) = -5 \quad x = -\frac{5}{m} + 3$

$$A(m) = \frac{1}{2}\left(-\frac{5}{m} + 3\right)(-3m + 5)$$

$$= \frac{1}{2}(15 - 25m^{-1} - 9m + 15)$$

$$= 15 - \frac{25}{2}m^{-1} - \frac{9}{2}m$$

(b) $A'(m) = \frac{25}{2}m^{-2} - \frac{9}{2} = 0$

$$\frac{25}{2}m^{-2} = \frac{9}{2}, \quad m^{-2} = \frac{9}{25} \qquad m^2 = \frac{25}{9}$$

$m = -\frac{5}{3}$ (negative!)

5. Evaluate

$$(a) \qquad \int_0^\pi \sqrt{x} + \sin x \, dx$$

$$(b) \qquad \int_{-2}^0 |x+1| \, dx$$

$$(a) \int_0^\pi x^{1/2} + \sin x \, dx = \frac{2}{3} x^{3/2} - \cos x \Big|_0^\pi$$

$$= \left(\frac{2}{3}\pi^{3/2} - \cos\pi\right) - (0 - \cos 0)$$

$$(b) \int_{-2}^0 |x+1| \, dx = \int_{-2}^{-1} -(x+1)dx + \int_{-1}^0 x+1 \, dx$$

$$= -\frac{1}{2}x^2 - x \Big|_{-2}^{-1} + \frac{1}{2}x^2 + x \Big|_{-1}^0$$

$$= \left(\left(-\frac{1}{2}(-1)^2 - (-1)\right) - \left(-\frac{1}{2}(-2)^2 - (-2)\right)\right)$$

$$-\left(0 - \left(\frac{1}{2}(-1)^2 + (-1)\right)\right)$$

$$= \left(-\frac{1}{2} + 1\right) - \left(-\frac{1}{2}(4) + 2\right) - \left(\frac{1}{2}(1) - 1\right) = 1$$

6. Find the value of x such that the tangent line to the curve $y = x^3 + 1$ at x passes through the point $(0, -1)$.

$y' = 3x^2$ $\qquad \frac{y-(-1)}{x-0} = 3x^2$

$y + 1 = 3x^3$

$(x^3+1)+1 = 3x^3 \qquad 2x^3 = 2$

$x^3 = 1 \qquad x = 1$

7. Evaluate the indefinite integrals

$$(a) \quad \int \frac{x}{(x^2+1)^2}\, dx$$

$$(b) \quad \int \sec^3 x \tan x \, dx$$

(a) $\int (x^2+1)^{-2} x \, dx$

$u = x^2 + 1 \quad du = 2x\,dx \quad x\,dx = \frac{1}{2} du$

$= \int u^{-2} \frac{1}{2} du = \frac{1}{2}(-u^{-1}) + C$

$= -\frac{1}{2}(x^2+1)^{-1} + C$

(b) $\int \sec^2 x \sec x \tan x \, dx$

$u = \sec x \quad du = \sec x \tan x \, dx$

$= \int u^2 \, du = \frac{1}{3} u^3 + C = \frac{1}{3} \sec^3 x + C$

8. Prove that if $f(x)$ is continuous on $[a, b]$ and $f(a) < f(b)$, then there exists c in $[a, b]$ such that

$$f(c) = \frac{f(a) + f(b)}{2}.$$

$$f(a) < \frac{f(a) + f(b)}{2} < f(b) \quad \text{because}$$

$$f(a) = \frac{f(a) + f(a)}{2} < \frac{f(a) + f(b)}{2} < \frac{f(b) + f(b)}{2} = f(b)$$

so c exists by the Intermediate Value Theorem

9. Find the value of a for which

$$\lim_{x\to 2} \frac{x^2 - 3x + a}{x - 2}$$

exists and evaluate the limit.

$$\begin{array}{r|l} & x - 1 \\ \hline x-2 & x^2 - 3x + a \\ & \underline{x^2 - 2x} \\ & \quad -x + a \\ & \quad \underline{-x + 2} \\ & \qquad a - 2 = 0 \end{array}$$

if $a = 2$ then

$$\lim_{x\to 2} \frac{x^2-3x+2}{x-2} = \lim_{x\to 2} \frac{\cancel{(x-2)}(x-1)}{\cancel{x-2}}$$

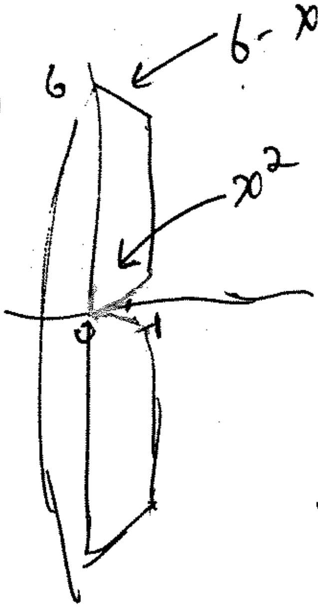
$$= \lim_{x\to 2} x - 1 = 1$$

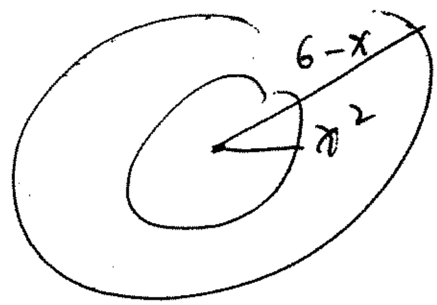
10. Let A be the region in the first quadrant bounded by the curves $x = 1, y = 6 - x$ and $y = x^2$. Write the volumes of the following solids as definite integrals but do NOT evaluate the integrals.

(a) The solid formed by rotating A about the line $y = 0$.

(b) The solid formed by rotating A about the line $x = 1$.

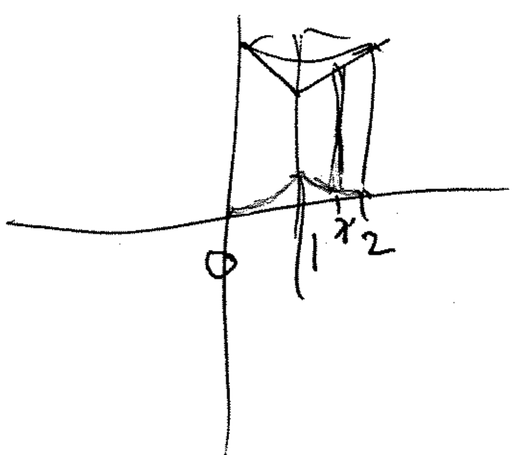
(a)





$$\int_0^1 \pi(6-x)^2 - \pi(x^2)^2 \, dx$$

(b)



$$\int_1^2 2\pi(x-1)((6-x)-x^2)\,dx$$