1. Find the (absolute) maximum and minimum values of

$$f(x) = (x^2 - 2)^{2/3}$$

on the interval $[-1, 3]$, showing your work.

$$f'(x) = \frac{2}{3}(x^2-2)^{-1/3}(2x) = \frac{4x}{3(x^2-2)}$$

Critical numbers: $-1, 0, \sqrt{2}, 3$

$f(-1) = (-1)^{2/3} = 1$

$f(0) = (-2)^{2/3} = \sqrt[3]{4}$

$f(\sqrt{2}) = 0$ min.

$f(3) = 7^{2/3} = \sqrt[3]{49}$ max

2. Find a general formula for $f^{(n)}(x)$ where

$$f(x) = \frac{1}{(2x+1)^3}.$$

Note: You may make use of a factorial symbol in your formula if you wish, but it is not required for full credit.

$f(x) = (2x+1)^{-3}$

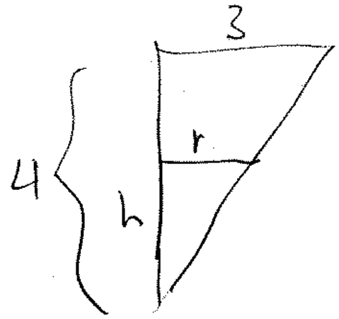
$f'(x) = -3(2x+1)^{-4}(2)$

$f''(x) = (-3)(-4)(2x+1)^{-5}(2)(2)$

$f^{(n)}(x) = (-1)^n(3)(4)\cdots(n+2)(2x+1)^{-(n+3)}(2)^n$

$[= (-1)^n (n+2)! \, (2x+1)^{-(n+3)} (2)^{n-1}]$

3. A tank is in the shape of an inverted cone of height $4m$ and radius $3m$ at the top. Water is being pumped into the tank at a constant rate of $7m^3/min$. However, water is also leaking out of the tank at a constant rate. If the depth of the water is rising at a rate of $8m/min$ when the water level is $\frac{2}{3}m$, at what rate is water leaking from the tank? Note: the volume of a cone of radius r and height h is $\frac{1}{3}\pi r^2 h$.



$$\frac{h}{r} = \frac{4}{3} \qquad r = \frac{3}{4}h$$

$$V = \frac{1}{3}\pi r^2 h = \frac{1}{3}\pi\left(\frac{3}{4}h\right)^2 h$$

$$= \frac{3\pi}{16}h^3$$

$$\frac{dV}{dt} = \frac{3\pi}{16}3h^2\frac{dh}{dt}$$

$$7 - C = \frac{9\pi}{16}\left(\frac{2}{3}\right)^2(8) = 2\pi$$

rate of leak $= C = 7 - 2\pi$ m^3/min

4. The function

$$f(x) = \sqrt{x^2 + 3x} + x$$

has a horizontal asymptote. Find it, showing your work.

$$\lim_{x \to -\infty} \sqrt{x^2+3x} + x = \lim_{x \to \infty} \sqrt{(-x)^2 + 3(-x)} + (-x)$$

$$= \lim_{x \to \infty} \sqrt{x^2 - 3x} - x \cdot \frac{\sqrt{x^2 - 3x} + x}{\sqrt{x^2 - 3x} + x}$$

$$= \lim_{x \to \infty} \frac{(x^2 - 3x) - x^2}{\sqrt{x^2 - 3x} + x}$$

$$= \lim_{x \to \infty} \frac{-3x}{\sqrt{x^2 - 3x} + x} \cdot \frac{\frac{1}{x}}{\frac{1}{x}}$$

$$= \lim_{x \to \infty} \frac{-3}{\sqrt{1 - \frac{3}{x}} + 1} = -\frac{3}{2}$$

5. Let

$$f(x) = ax^3 + bx^2 + cx + d$$

be the general cubic polynomial where a, b, c and d are constants and $a \neq 0$.

(10 points) (a) Prove that $f(x)$ has a point of inflection.

(10 points) (b) Prove that if $x = 0$ and $x = 1$ are critical numbers of $f(x)$, then the point of inflection is $x = \frac{1}{2}$.

(a) $f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b = 0$

$x = -b/3a$ is a point of inflection

(b) $f'(0) = c = 0$

$f'(1) = 3a(1) + 2b(1) + 0 = 0$

$b = -\frac{3}{2}a$

so the point of inflection is

$$x = -\frac{b}{3a} = -\frac{\left(-\frac{3}{2}a\right)}{3a} = \frac{1}{2}$$

(part (a))