1. Differentiate

(7 points) (a) $$f(x) = \frac{\sqrt{x}}{1+x^2}$$

(13 points) (b) $$f(x) = \csc^2(\cos x)$$

(a) $$f'(x) = \frac{\frac{1}{2}x^{-1/2}(1+x^2) - x^{1/2}(2x)}{(1+x^2)^2}$$

(b) $$f'(x) = 2\csc(\cos x)(-\csc(\cos x)\cot(\cos x))(-\sin x)$$

2. Suppose $f(x)$ is a one-to-one function with the property

$$f'(x) = 1 + (f(x))^2$$

for all x. Let $g(x) = f^{-1}(x)$ be the inverse of $f(x)$. Show that

$$g'(x) = \frac{1}{1+x^2}.$$

$$g'(x) = \frac{1}{f'(g(x))}$$

$$f'(g(x)) = 1 + (f(g(x)))^2 = 1 + x^2$$

$$g'(x) = \frac{1}{1+x^2}$$

3. There are two values of x for which the tangent to the curve

$$x^2 - xy + 2y^2 = 3$$

at (x, y) is horizontal. Find those values of x.

$$2x - y - x\frac{dy}{dx} + 4y\frac{dy}{dx} = 0$$

$$(4y - x)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y - 2x}{4y - x} = 0 \quad \text{if} \quad y = 2x$$

$$x^2 - x(2x) + 2(2x)^2 = 3$$

$$7x^2 = 3$$

$$x = \pm\sqrt{\frac{3}{7}}$$

4. Use the **definition** of the derivative to show that if

$$g(x) = xf(x)$$

then

$$g'(x) = f(x) + xf'(x).$$

$$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

$$= \lim_{h \to 0} \frac{(x+h)f(x+h) - xf(x)}{h}$$

$$= \lim_{h \to 0} \frac{xf(x+h) + hf(x+h) - xf(x)}{h}$$

$$= \lim_{h \to 0} \frac{\cancel{h}f(x+h)}{\cancel{h}} + x\,\frac{f(x+h) - f(x)}{h}$$

$$= \lim_{h \to 0} f(x+h) + \lim_{h \to 0} x \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

$$= f(x) + xf'(x)$$

5. There is a number a for which

$$\lim_{x\to 2} \frac{ax^3 - 7x^2 + ax - 2}{x-2}$$

exists. Find the value of a and then evaluate the limit.

$$a(2)^3 - 7(2)^2 + a(2) - 2 = 0$$

$$10a - 30 = 0 \qquad a = 3$$

$$\begin{array}{r|l} & 3x^2 - x + 1 \\ \hline x-2 & 3x^3 - 7x^2 + 3x - 2 \\ & \underline{3x^3 - 6x^2} \\ & \quad -x^2 + 3x \\ & \quad \underline{-x^2 + 2x} \\ & \qquad x - 2 \end{array}$$

$$\lim_{x\to 2} \frac{3x^3 - 7x^2 + 3x - 2}{x-2}$$

$$= \lim_{x\to 2} \frac{\cancel{(x-2)}(3x^2 - x + 1)}{\cancel{x-2}}$$

$$= 3(2)^2 - (2) + 1 = 11$$