

1. Find the critical numbers and the intervals of increase and decrease of the function

$$f(x) = \frac{x^2}{x+8}$$

$$f'(x) = \frac{2x(x+8) - x^2}{(x+8)^2} = \frac{x^2 + 16x}{(x+8)^2} = \frac{x(x+16)}{(x+8)^2}$$

Critical numbers: $x = 0, -16, -8$

	x	$x+16$	$f'(x)$	
$(-\infty, -16)$	-	-	+	increasing
$(-16, -8)$	-	+	-	decreasing
$(-8, 0)$	-	+	-	decreasing
$(0, \infty)$	+	+	+	increasing

2. Let A be the region of the first quadrant bounded by the curves $y = \cos x$ and $x = \pi/4$. Write the volumes of the following solids as definite integrals but do not evaluate the integrals.

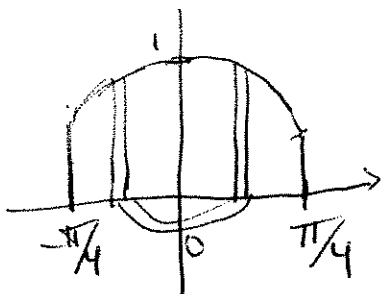
(10 points)

(a) The solid formed by rotating A about the line $x = 0$.

(10 points)

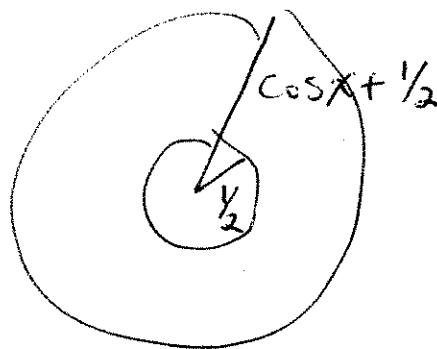
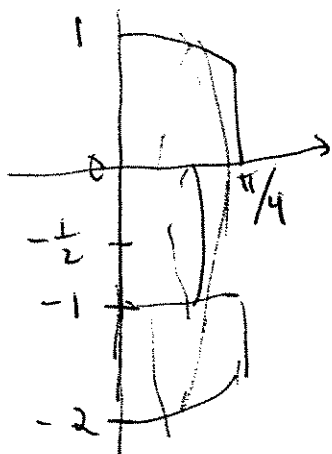
(b) The solid formed by rotating A about the line $y = -1/2$.

(a)



$$\int_0^{\pi/4} 2\pi x \cos x \, dx$$

(b)



$$\int_0^{\pi/4} \pi \left(\cos x + \frac{1}{2} \right)^2 - \pi \left(\frac{1}{2} \right)^2 \, dx$$

3. Differentiate

(10 points) (a) $f(x) = \tan\left(\frac{1}{2}\sin(x^3 + 2x)\right)$

(10 points) (b) $f(x) = \int_x^{x^3} \frac{t^{1/3}}{t^{2/3} + 1} dt$

$$\begin{aligned} \text{(a) } f'(x) &= \sec^2\left(\frac{1}{2}\sin(x^3 + 2x)\right) \frac{d}{dx}\left(\frac{1}{2}\sin(x^3 + 2x)\right) \\ &= \sec^2\left(\frac{1}{2}\sin(x^3 + 2x)\right) \frac{1}{2} \cos(x^3 + 2x) (3x^2 + 2) \end{aligned}$$

$$\begin{aligned} \text{(b) } f(x) &= \int_x^0 \frac{t^{1/3}}{t^{2/3} + 1} dt + \int_0^{x^3} \frac{t^{1/3}}{t^{2/3} + 1} dt \\ &= -\int_0^x \frac{t^{1/3}}{t^{2/3} + 1} dt + \int_0^{x^3} \frac{t^{1/3}}{t^{2/3} + 1} dt \end{aligned}$$

$$f'(x) = -\frac{x^{1/3}}{x^{2/3} + 1} + \frac{(x^3)^{1/3}}{(x^3)^{2/3} + 1} (3x^2)$$

(5 points)

4. (a) In the space just below, write in mathematical symbols the meaning of " $f(x)$ is continuous at $x = a$ ".

$$\lim_{x \rightarrow a} f(x) = f(a)$$

(15 points)

(b) The meaning of " $f(x)$ is differentiable at $x = a$ " is that the limit

$$\lim_{x \rightarrow a} \frac{f(x) - f(a)}{x - a}$$

exists. Since the limit of the denominator is zero and the limit of the quotient exists, the limit of the numerator must also be zero. Complete the proof that differentiability implies continuity by using properties of limits to prove that if

$$\lim_{x \rightarrow a} (f(x) - f(a)) = 0$$

then $f(x)$ is continuous at $x = a$.

$$\begin{aligned} 0 &= \lim_{x \rightarrow a} (f(x) - f(a)) \\ &= \lim_{x \rightarrow a} f(x) - \lim_{x \rightarrow a} f(a) \\ &= \lim_{x \rightarrow a} f(x) - f(a) \end{aligned}$$

$$\lim_{x \rightarrow a} f(x) = f(a)$$

5. Find the average value of the function

$$f(x) = x\sqrt{2x-3}$$

on the interval $[2, 6]$.

$$\text{Average Value} = \frac{1}{6-2} \int_2^6 x\sqrt{2x-3} \, dx$$

$$\begin{aligned} u &= 2x-3 & du &= 2 \, dx & u(2) &= 1 \\ x &= \frac{1}{2}(u+3) & dx &= \frac{1}{2} \, du & u(6) &= 9 \end{aligned}$$

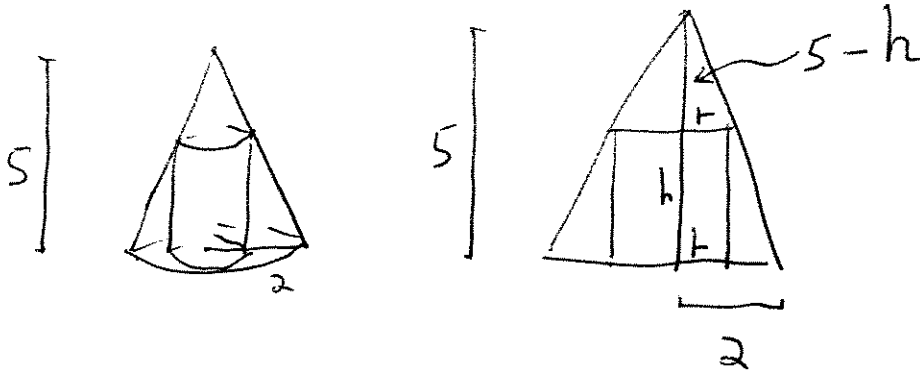
$$= \frac{1}{4} \int_1^9 \frac{1}{2}(u+3) u^{\frac{1}{2}} \frac{1}{2} \, du$$

$$= \frac{1}{16} \int_1^9 u^{\frac{3}{2}} + 3u^{\frac{1}{2}} \, du$$

$$= \frac{1}{16} \left[\frac{2}{5} u^{\frac{5}{2}} + 2u^{\frac{3}{2}} \right]_1^9$$

$$= \frac{1}{16} \left[\left(\frac{2}{5} (9)^{\frac{5}{2}} + 2(9)^{\frac{3}{2}} \right) - \left(\frac{2}{5} + 2 \right) \right]$$

6. Find the radius and height of the right circular cylinder of largest volume that can be inscribed in a cone whose height is 5 m and base radius is 2 m.



$$\frac{5-h}{r} = \frac{5}{2} \quad 5-h = \frac{5}{2}r \quad h = 5 - \frac{5}{2}r$$

$$V = \pi r^2 h = \pi r^2 \left(5 - \frac{5}{2}r\right) = 5\pi r^2 - \frac{5}{2}\pi r^3$$

$$V'(r) = 10\pi r - \frac{15}{2}\pi r^2 = 0$$

$$r \left(10\pi - \frac{15}{2}\pi r\right) = 0 \quad 10\pi - \frac{15}{2}\pi r = 0$$

$$\frac{15}{2}r = 10 \quad r = \frac{20}{15} = \frac{4}{3}$$

$$h = 5 - \frac{5}{2} \left(\frac{4}{3}\right) = 5 - \frac{10}{3} = \frac{5}{3}$$

7. Show that the following inequalities are true.

(15 points) (a) $x\sqrt{x-x^2} \leq \frac{3\sqrt{3}}{16}$

(Hint: What is the maximum of that function?)

(5 points) (b) $\int_1^2 \sqrt{1+x^4} dx \geq \frac{7}{3}$

(a) $f(x) = x(x-x^2)^{1/2}$

$$f'(x) = (x-x^2)^{1/2} + x \frac{1}{2} (x-x^2)^{-1/2} (1-2x)$$

$$= (x-x^2)^{1/2} + \frac{x(1-2x)}{2(x-x^2)^{1/2}}$$

$$= \frac{2(x-x^2) + x - 2x^2}{2(x-x^2)^{1/2}} = \frac{3x - 4x^2}{2(x-x^2)^{1/2}} = 0$$

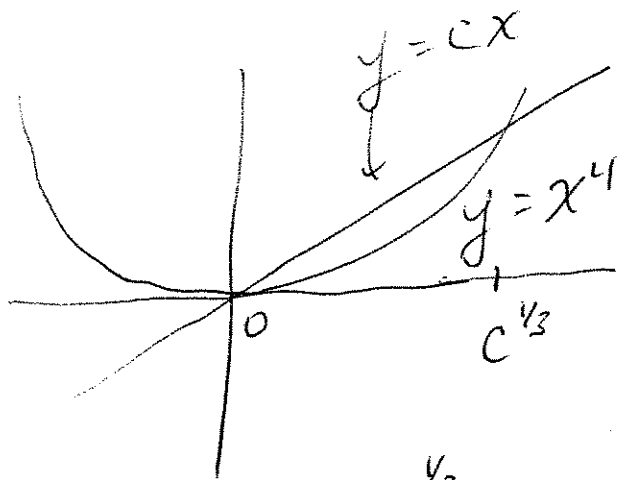
$$3 - 4x = 0 \quad x = 3/4$$

$$f(x) \leq f\left(\frac{3}{4}\right) = \left(\frac{3}{4}\right) \sqrt{\frac{3}{4} - \frac{9}{16}} = \frac{3}{4} \sqrt{\frac{3}{16}} = \frac{3\sqrt{3}}{16}$$

(b) $\int_1^2 \sqrt{1+x^4} dx \geq \int_1^2 \sqrt{x^4} dx$

$$= \int_1^2 x^2 dx = \left. \frac{1}{3} x^3 \right|_1^2 = \frac{7}{3}$$

8. There is a number $c > 0$ such that the area between the curve $y = x^4$ and the line $y = cx$ is $96/10$. Find the value of c .



$$x^4 = cx$$

$$x(x^3 - c) = 0$$

$$x^3 = c \quad x = c^{1/3}$$

$$\frac{96}{10} = \int_0^{c^{1/3}} cx - x^4 dx = \left. \frac{c}{2} x^2 - \frac{1}{5} x^5 \right|_0^{c^{1/3}}$$

$$= \frac{c}{2} (c^{2/3}) - \frac{1}{5} c^{5/3} = c^{5/3} \left(\frac{1}{2} - \frac{1}{5} \right)$$

$$= c^{5/3} \left(\frac{3}{10} \right) \quad c^{5/3} = \left(\frac{10}{3} \right) \frac{96}{10} = 32$$

$$c = (32)^{3/5} = (2^5)^{3/5} = 8$$

9. The equations below define curves y . For each equation, calculate the value of the second derivative y'' when $x = 0$.

(7 points) (a) $y = \int_{-1}^x \frac{3t}{t^3+1} dt$

(13 points) (b) $\sin y = x^2$

$$(a) \quad y' = \frac{3x}{x^3+1} \quad y'' = \frac{3(x^3+1) - 3x(3x^2)}{(x^3+1)^2}$$

if $x=0$ then $y'' = 3$

(b) $\cos y \frac{dy}{dx} = 2x$

$$\frac{dy}{dx} = \frac{2x}{\cos y} \quad \frac{d^2y}{dx^2} = \frac{2(\cos y) - 2x(-\sin y) \frac{dy}{dx}}{(\cos y)^2}$$

$$= \frac{2(\cos y)}{(\cos y)^2} = \frac{2}{\cos y} \quad \text{when } x=0$$

(Note $\sin y = (0)^2$ so $\cos y = \pm 1$)

10. Suppose $f(x)$ is continuous on $[1, 4]$ with $f(1) = -5$ and also $f'(x) \geq 2$ for all x in $(1, 4)$.

(7 points) (a) Prove that there is at most one x in $[1, 4]$ such that $f(x) = 0$.

(13 points)

(b) Prove that there does exist x in $[1, 4]$ such that $f(x) = 0$.
(Hint: What can you conclude about $f(4) - f(1)$?)

(a) If $f(a) = f(b) = 0$ for some $a < b$ in $[1, 4]$ then, by Rolle's Theorem, $f'(c) = 0$ for some c in (a, b) , but $f'(x) \geq 2$.

(b) By the Mean Value Theorem

$$f(4) - f(1) = f'(c)(4-1)$$

for some c in $(1, 4)$. Since $f(1) = -5$ and $f'(c) \geq 2$ then

$$f(4) - (-5) = f'(c)(3) \geq 6 \quad \text{so}$$

$f(4) \geq 1 > 0$ while $f(1) = -5 < 0$ so $f(x) = 0$ for some x in $(1, 4)$ by the Intermediate Value Theorem.