1. Given the function

$$f(x) = x^4 - 2x^3 - 12x^2 + 5$$

determine where the function is concave upward and concave downward, and find the points of inflection, if there are any.

$$f'(x) = 4x^3 - 6x^2 - 24x$$

$$f''(x) = 12x^2 - 12x - 24$$

$$= 12(x^2 - x - 2)$$

$$= 12(x-2)(x+1)$$

Interval	$x-2$	$x+1$	$f''(x)$	Concave
$(-\infty, -1)$	neg	neg	pos	upward
$(-1, 2)$	neg	pos	neg	downward
$(2, \infty)$	pos	pos	pos	upward

points of inflection: $x = -1, 2$

2. Consider the function

$$f(x) = \frac{\sqrt{2x^2 - 1}}{x}.$$

(a) Determine the domain of $f(x)$.

(b) Find the horizontal asymptotes of $f(x)$, if there are any.

(a) $x \neq 0 \quad 2x^2 - 1 \geq 0 \quad x^2 \geq \frac{1}{2}$

so $x \geq \sqrt{\frac{1}{2}}$ or $x \leq -\sqrt{\frac{1}{2}}$

Domain: $(-\infty, -\sqrt{\frac{1}{2}}]$ and $[\sqrt{\frac{1}{2}}, \infty)$

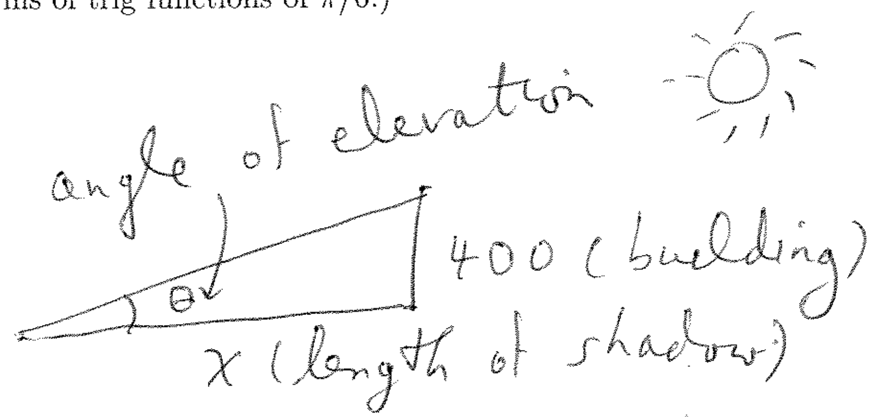
(b) $$\lim_{x \to \infty} \frac{\sqrt{2x^2-1}}{x} = \lim_{x \to \infty} \frac{\sqrt{2x^2-1}}{\sqrt{x^2}} = \lim_{x \to \infty} \sqrt{\frac{2x^2-1}{x^2}}$$

$$= \lim_{x \to \infty} \sqrt{2 - \frac{1}{x^2}} = \sqrt{2}$$

$$\lim_{x \to -\infty} \frac{\sqrt{2x^2-1}}{x} = \lim_{x \to \infty} \frac{\sqrt{2(-x)^2-1}}{-x}$$

$$= -\lim_{x \to \infty} \frac{\sqrt{2x^2-1}}{x} = -\sqrt{2}$$

3. As the Sun sets, the angle of elevation of the Sun above the horizon is decreasing at the rate of $\frac{1}{4}$ radian/hr. How fast is the shadow cast by a 400-foot-tall building increasing when the angle of elevation of the Sun is $\pi/6$ radians. (You can leave your answer in terms of trig functions of $\pi/6$.)



Note x and θ depend on time (as sun moves)

$$\cot\theta = \frac{x}{400} \qquad \frac{d\theta}{dt} = -\frac{1}{4}$$

$$x = 400\cot\theta$$

$$\frac{dx}{dt} = 400(-\csc^2\theta)\frac{d\theta}{dt}$$

$$= 400\left(-\csc^2\left(\frac{\pi}{6}\right)\right)\left(-\frac{1}{4}\right)$$

4. Find all the local maxima and local minima of the function

$$f(x) = x^{2/3}(1-x)^2.$$

$$f'(x) = \frac{2}{3}x^{-1/3}(1-x)^2 + x^{2/3}\,2(1-x)(-1)$$
$$= 2x^{-1/3}(1-x)\left(\frac{1}{3}(1-x) - x\right)$$
$$= \frac{2}{3}x^{-1/3}(1-x)(1-x-3x)$$
$$= \frac{2}{3}x^{-1/3}(1-x)(1-4x)$$

critical numbers: $x = 0$ (no derivative), 1, $1/4$

Interval	$x^{-1/3}$	$1-x$	$1-4x$	$f'(x)$
$(-\infty, 0)$	neg	pos	pos	neg
$(0, 1/4)$	pos	pos	pos	pos
$(1/4, 1)$	pos	pos	neg	neg
$(1, \infty)$	pos	neg	neg	pos

local min at $x = 0, 1$

local max at $x = 1/4$

5. Suppose that a function $f(x)$ has second derivative $f''(x)$ for all real numbers x.

(a) Prove that there exists a point c in $(0, 1)$ such that

$$f(1) - f(0) = f'(c).$$

(b) Prove that if $|f''(x)| \leq 1$ for all x in $(0, 1)$, then

$$|f(1) - f(0) - f'(0)| < 1.$$

(a) By the Mean Value Theorem

$$f(1) - f(0) = f'(c)(1-0) = f'(c)$$

(b) By part (a)

$$f(1) - f(0) - f'(0) = f'(c) - f'(0)$$

By the Mean Value Theorem, there is d in $(0,c)$ so that

$$f'(c) - f'(0) = f''(d)(c-0)$$

$$|f(1) - f(0) - f'(0)|$$

$$= |f'(c) - f'(0)| = |f''(d)(c)|$$

$$= |f''(d)||c| < 1$$

because $|f''(d)| \leq 1$ and $c < 1$.