

1. Let

$$f(x) = \frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{3}{2}x^2 + 1.$$

(a) Find the critical numbers of f .

(b) Find the intervals of increase and decrease of f .

$$(a) f'(x) = x^3 - 2x^2 - 3x = 0$$

$$x(x^2 - 2x - 3) = 0$$

$$x(x-3)(x+1) = 0$$

$$x = 0, 3, -1$$

(b)

Interval

$(-\infty, -1)$

$(-1, 0)$

$(0, 3)$

$(3, \infty)$

x

-

-

+

+

$x-3$

-

-

-

+

$x+1$

-

+

+

+

$f'(x)$

-

+

-

+

dec.

increas.

dec.

increas.

2. Calculate the limit, if it exists.

$$\lim_{x \rightarrow 0} \frac{\tan 4x}{x}$$

$$\lim_{x \rightarrow 0} \frac{\tan 4x}{x} = \lim_{x \rightarrow 0} \frac{1}{x} \frac{\sin 4x}{\cos 4x}$$

$$= \lim_{x \rightarrow 0} \frac{4}{\cos 4x} \frac{\sin 4x}{4x}$$

$$= \lim_{x \rightarrow 0} \frac{4}{\cos 4x} \lim_{4x \rightarrow 0} \frac{\sin 4x}{4x} = \left(\frac{4}{1}\right)(1) = 4$$

3. Calculate

(a) the slope of the tangent to the curve

$$x \sin y = x + 1$$

at $(-1, 0)$.

(b) the derivative of

$$f(x) = \int_{-1}^{x^2} \sqrt{1+t^4} dt$$

at $x = 2$.

$$(a) \quad \sin y + x \cos y \frac{dy}{dx} = 1$$

$$\sin 0 + (-1) \cos 0 \frac{dy}{dx} = 1$$

$$-\frac{dy}{dx} = 1 \quad \frac{dy}{dx} = -1$$

$$(b) \quad g(x) = \int_{-1}^x \sqrt{1+t^4} dt \quad g'(x) = \sqrt{1+x^4}$$

$$f(x) = g(x^2) \quad f'(x) = g'(x^2)(2x)$$

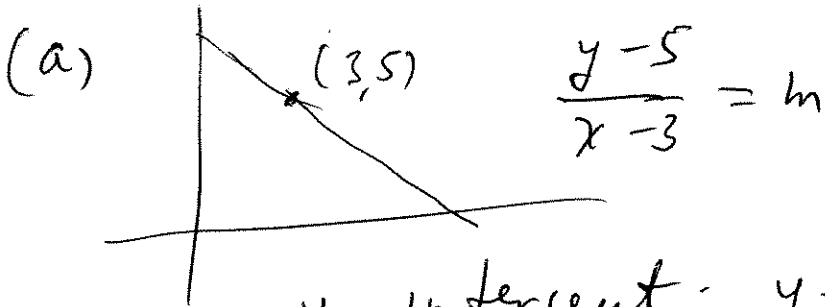
$$= \sqrt{1+(x^2)^4} (2x)$$

$$f'(2) = \sqrt{1+(4)^4} (4)$$

4. A line of negative slope through $(3, 5)$ bounds a triangular region of the first quadrant of the plane.

(a) Write the area of the triangle as a function of the slope of the line.

(b) Find the slope that makes the area of the triangular region as small as possible.



$$y\text{-intercept: } y-5 = -3m \quad y = -3m + 5$$

$$x\text{-intercept: } m(x-3) = -5 \quad x = -\frac{5}{m} + 3$$

$$A(m) = \frac{1}{2} \left(-\frac{5}{m} + 3\right) (-3m + 5)$$

$$= \frac{1}{2} (15 - 25m^{-1} - 9m + 15)$$

$$= 15 - \frac{25}{2} m^{-1} - \frac{9}{2} m$$

$$(b) \quad A'(m) = \frac{25}{2} m^{-2} - \frac{9}{2} = 0$$

$$\frac{25}{2} m^{-2} = \frac{9}{2} \quad m^{-2} = \frac{9}{25}$$

$$m^2 = \frac{25}{9}$$

$$m = -\frac{5}{3} \text{ (negative!)}$$

5. Evaluate

$$(a) \int_0^{\pi} \sqrt{x} + \sin x \, dx$$

$$(b) \int_{-2}^0 |x+1| \, dx$$

$$(a) \int_0^{\pi} x^{\frac{1}{2}} + \sin x \, dx = \frac{2}{3} x^{\frac{3}{2}} - \cos x \Big|_0^{\pi}$$
$$= \left(\frac{2}{3} \pi^{\frac{3}{2}} - \cos \pi \right) - (0 - \cos 0)$$

$$(b) \int_{-2}^0 |x+1| \, dx = \int_{-2}^{-1} -(x+1) \, dx + \int_{-1}^0 x+1 \, dx$$

$$= -\frac{1}{2} x^2 - x \Big|_{-2}^{-1} + \frac{1}{2} x^2 + x \Big|_{-1}^0$$

$$= \left(\left(-\frac{1}{2} (-1)^2 - (-1) \right) - \left(-\frac{1}{2} (-2)^2 - (-2) \right) \right) - \left(0 - \left(\frac{1}{2} (-1)^2 + (-1) \right) \right)$$

$$= \left(-\frac{1}{2} + 1 \right) - \left(-\frac{1}{2} (4) + 2 \right) - \left(\frac{1}{2} (1) - 1 \right) = 1$$

6. Find the value of x such that the tangent line to the curve $y = x^3 + 1$ at x passes through the point $(0, -1)$.

$$y' = 3x^2 \quad \frac{y - (-1)}{x - 0} = 3x^2$$

$$y + 1 = 3x^3$$

$$(x^3 + 1) + 1 = 3x^3 \quad 2x^3 = 2$$

$$x^3 = 1 \quad x = 1$$

7. Evaluate the indefinite integrals

$$(a) \int \frac{x}{(x^2+1)^2} dx$$

$$(b) \int \sec^3 x \tan x dx$$

$$(a) \int (x^2+1)^{-2} x dx$$

$$u = x^2 + 1 \quad du = 2x dx \quad x dx = \frac{1}{2} du$$

$$= \int u^{-2} \frac{1}{2} du = \frac{1}{2} (-u^{-1}) + C$$

$$= -\frac{1}{2} (x^2+1)^{-1} + C$$

$$(b) \int \sec^2 x \sec x \tan x dx$$

$$u = \sec x \quad du = \sec x \tan x dx$$

$$= \int u^2 du = \frac{1}{3} u^3 + C = \frac{1}{3} \sec^3 x + C$$

8. Prove that if $f(x)$ is continuous on $[a, b]$ and $f(a) < f(b)$, then there exists c in $[a, b]$ such that

$$f(c) = \frac{f(a) + f(b)}{2}.$$

$$f(a) < \frac{f(a) + f(b)}{2} < f(b) \quad \text{because}$$

$$\begin{aligned} f(a) &= \frac{f(a) + f(a)}{2} < \frac{f(a) + f(b)}{2} \\ &< \frac{f(b) + f(b)}{2} = f(b) \end{aligned}$$

so c exists by the Intermediate Value Theorem

9. Find the value of a for which

$$\lim_{x \rightarrow 2} \frac{x^2 - 3x + a}{x - 2}$$

exists and evaluate the limit.

$$x-2 \overline{) \begin{array}{r} x-1 \\ x^2-3x+a \end{array}}$$

$$\begin{array}{r} x^2-2x \\ -x+a \end{array}$$

$$\begin{array}{r} -x+2 \\ a-2 \end{array}$$

$$a-2 = 0 \quad \text{if } a=2 \text{ then}$$

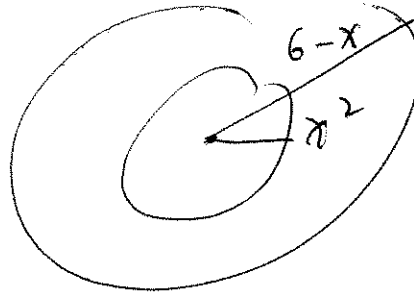
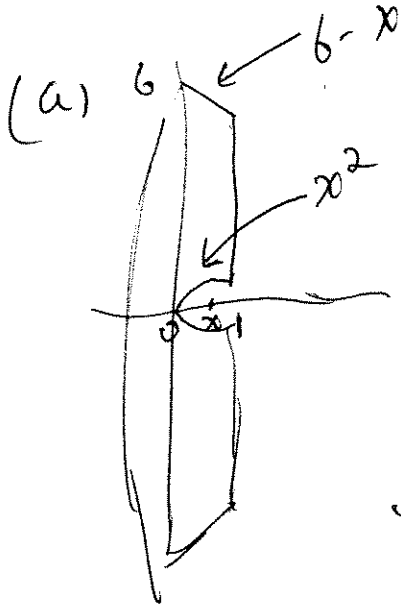
$$\lim_{x \rightarrow 2} \frac{x^2 - 3x + 2}{x - 2} = \lim_{x \rightarrow 2} \frac{\cancel{(x-2)}(x-1)}{\cancel{x-2}}$$

$$= \lim_{x \rightarrow 2} x - 1 = 1$$

10. Let A be the region in the first quadrant bounded by the curves $x = 1$, $y = 6 - x$ and $y = x^2$. Write the volumes of the following solids as definite integrals but do NOT evaluate the integrals.

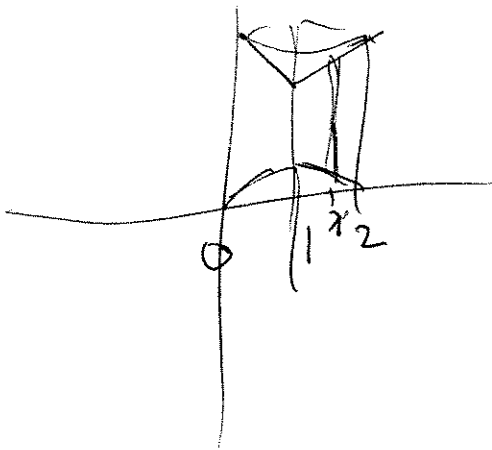
(a) The solid formed by rotating A about the line $y = 0$.

(b) The solid formed by rotating A about the line $x = 1$.



$$\int_0^1 \pi (6-x)^2 - \pi (x^2)^2 dx$$

(b)



$$\int_1^2 2\pi (x-1)((6-x)-x^2) dx$$