1. Given the function

$$f(x) = \frac{x}{x-2} \qquad (x \neq 2),$$

determine where the function is concave upward and where it is concave downward.

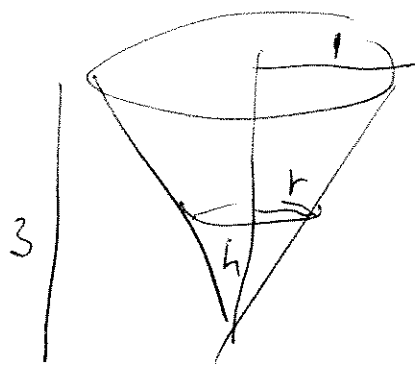
$$f'(x) = \frac{x - 2 - x}{(x-2)^2} = -2(x-2)^{-2}$$

$$f''(x) = 4(x-2)^{-3} = \frac{4}{(x-2)^3}$$

pos for $x > 2$ - concave up

neg for $x < 2$ - concave down

2. A paper cup in the shape of a cone is of height 3 in. and radius (at the top) of 1 in. Water is poured into the cup at a rate of $\frac{1}{3}$ in^3/s. How fast is the water level rising when the water is 2 in. deep? (Note: The volume of a cone of height h and radius r is $V = \frac{1}{3}\pi r^2 h$.)



$$\frac{r}{h} = \frac{1}{3} \qquad r = \frac{1}{3} h$$

$$V(h) = \frac{1}{3}\pi\left(\frac{1}{3}h\right)^2 h$$

$$= \frac{\pi}{27} h^3$$

$$\frac{1}{3} = \frac{dV}{dt} = \frac{\pi}{9} h^2 \frac{dh}{dt} \qquad \frac{dh}{dt} = \frac{3}{4\pi}$$

$$\frac{1}{3} = \frac{\pi}{9}(4)\frac{dh}{dt}$$

3. Find all the local maxima and local minima of

$$f(x) = (x^2 - 1)^{2/3}.$$

$$f'(x) = \frac{2}{3}(x^2-1)^{-1/3}(2x) = \frac{4}{3}\,\frac{x}{(x^2-1)^{1/3}}$$

Critical numbers: $x = 0$ and $+1, -1$
(no derivative)

Interval	x	$(x^2-1)^{1/3}$	$f'(x)$
$(-\infty, -1)$	neg	pos	neg
$(-1, 0)$	neg	neg	pos
$(0, 1)$	pos	neg	neg
$(1, \infty)$	pos	pos	pos

local min at $+1, -1$
local max at 0

4. A soft drink can in the shape of a right circular cylinder must contain 32 cubic inches of liquid. The metal used for the bottom of the can is the same as that used for the side but the metal used for the top costs 3 times as much as the metal used for the rest of the can. Find the dimensions (its height and radius of the base) of the can that will minimize the cost of the metal used in making it. (Note: You do not have to verify by means of a test that your answer is the minimum.)

$$V = \pi r^2 h = 32 \qquad h = \frac{32}{\pi r^2}$$

cost of side metal $= c$

$$\text{Cost} = 2\pi r h c + \pi r^2 c + \pi r^2 (3c)$$

$$C(r) = 2\pi r\left(\frac{32}{\pi r^2}\right)c + 4c\pi r^2$$

$$C(r) = 64r^{-1}c + 4c\pi r^2$$

$$C'(r) = -64r^{-2}c + 8c\pi r = 0$$

$$-64 + 8\pi r^3 = 0$$

$$r^3 = \frac{8}{\pi} \qquad r = \frac{2}{\sqrt[3]{\pi}} \qquad h = \frac{32}{\pi\left(\frac{2}{\sqrt[3]{\pi}}\right)^2}$$

5. Use the Mean Value Theorem to prove that, if $b > 0$, then

$$\sqrt{1+b} < 1 + \frac{b}{2}.$$

$f(x) = \sqrt{1+x}$ $\quad$ $f'(x) = \frac{1}{2}(1+x)^{-1/2}$

$= (1+x)^{1/2}$ $\quad$ $= \frac{1}{2\sqrt{1+x}}$

Mean Value Theorem $(a = 0)$ $\quad$ $0 < c < b$

$$\sqrt{1+b} - \sqrt{1+0} = \frac{1}{2\sqrt{1+c}}(b-0)$$

$$\sqrt{1+b} - 1 = \frac{b}{2\sqrt{1+c}} < \frac{b}{2}$$

$$\sqrt{1+b} < 1 + \frac{b}{2}$$