

1. Find dy/dx .

(a) $y = x \tan y$

(b) $y = \int_{\cos x}^1 \sqrt{1-t^2} dt$

(a) $\frac{dy}{dx} = \tan y + x \sec^2 y \frac{dy}{dx}$

$$(1 - x \sec^2 y) \frac{dy}{dx} = \tan y$$

$$\frac{dy}{dx} = \frac{\tan y}{1 - x \sec^2 y}$$

(b) $y = - \int_1^{\cos x} \sqrt{1-t^2} dt = -g(\cos x)$

where $g(x) = \int_1^x \sqrt{1-t^2} dt$

$$\frac{dy}{dx} = -g'(\cos x)(-\sin x) = \sqrt{1-\cos^2 x} (\sin x)$$

2. Given that

$$f''(x) = \frac{1+3x^4}{x^3}$$

$f(1) = 1$ and $f'(1) = 3$, find $f(x)$.

$$f''(x) = x^{-3} + 3x$$

$$f'(x) = -\frac{1}{2}x^{-2} + \frac{3}{2}x^2 + C$$

$$3 = f'(1) = -\frac{1}{2} + \frac{3}{2} + C \quad C = 2$$

$$f'(x) = -\frac{1}{2}x^{-2} + \frac{3}{2}x^2 + 2$$

$$f(x) = \frac{1}{2}x^{-1} + \frac{1}{2}x^3 + 2x + D$$

$$1 = f(1) = \frac{1}{2} + \frac{1}{2} + 2 + D \quad D = -2$$

$$f(x) = \frac{1}{2}x^{-1} + \frac{1}{2}x^3 + 2x - 2$$

3. Find the absolute maximum and absolute minimum of the function

$$f(x) = x\sqrt{1-4x}$$

on the interval $[-1/4, 1/4]$.

$$f(x) = x(1-4x)^{\frac{1}{2}}$$

$$f'(x) = (1-4x)^{\frac{1}{2}} + x\left(\frac{1}{2}\right)(1-4x)^{-\frac{1}{2}}(-4)$$

$$= (1-4x)^{\frac{1}{2}} - \frac{2x}{(1-4x)^{\frac{1}{2}}}$$

$$= \frac{1-4x-2x}{(1-4x)^{\frac{1}{2}}} = \frac{1-6x}{(1-4x)^{\frac{1}{2}}}$$

$$= 0 \text{ if } x = \frac{1}{6}$$

$$f(-\frac{1}{4}) = -\frac{1}{4}\sqrt{2} \quad \text{absolute minimum}$$

$$f(\frac{1}{6}) = \frac{1}{6}\sqrt{\frac{1}{3}} \quad \text{absolute maximum}$$

$$f(\frac{1}{4}) = 0$$

4. Evaluate the definite integrals.

$$(a) \int_0^4 |\sqrt{x} - 1| dx$$

$$(b) \int_0^1 x\sqrt{3-x^2} dx$$

$$(a) \quad |\sqrt{x} - 1| = \begin{cases} x^{\frac{1}{2}} - 1 & x^{\frac{1}{2}} \geq 1 \\ 1 - x^{\frac{1}{2}} & 0 \leq x^{\frac{1}{2}} < 1 \end{cases}$$

$$\int_0^1 1 - x^{\frac{1}{2}} dx + \int_1^4 x^{\frac{1}{2}} - 1 dx$$

$$= x - \frac{2}{3} x^{\frac{3}{2}} \Big|_0^1 + \frac{2}{3} x^{\frac{3}{2}} - x \Big|_1^4$$

$$= \frac{1}{3} + \left(\frac{2}{3} (4)^{\frac{3}{2}} - 4 \right) - \left(\frac{2}{3} - 1 \right) (= 2)$$

$$(b) \quad u = 3 - x^2 \quad du = -2x dx \quad x dx = -\frac{1}{2} du$$

$$u(0) = 3 \quad u(1) = 2$$

$$\left(-\frac{1}{2}\right) \int_3^2 u^{\frac{1}{2}} du = \frac{1}{2} \int_2^3 u^{\frac{1}{2}} du$$

$$= \frac{1}{2} \left(\frac{2}{3} \right) u^{\frac{3}{2}} \Big|_2^3 = \frac{1}{3} (3^{\frac{3}{2}} - 2^{\frac{3}{2}})$$

5. Calculate the limits, using only techniques from Math 31A.

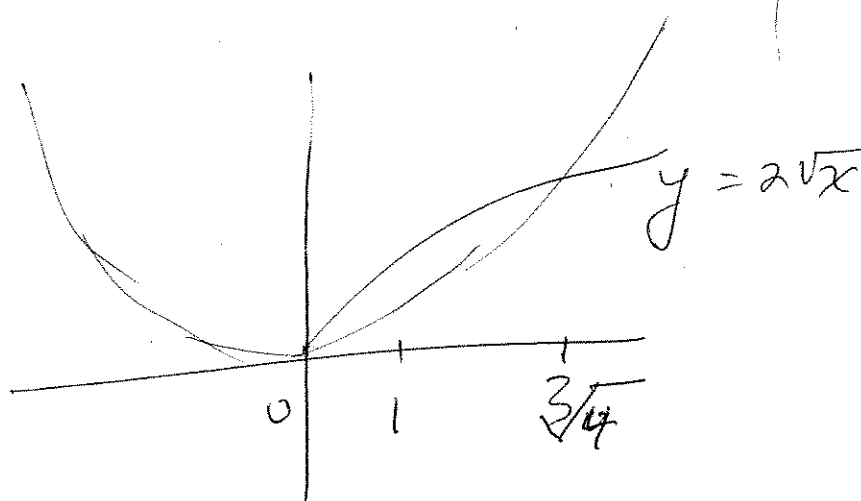
$$(a) \quad \lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin x}$$

$$(b) \quad \lim_{x \rightarrow -\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1}$$

$$\begin{aligned} (a) \quad \lim_{x \rightarrow 0} \frac{\cos x - 1}{\sin x} &= \lim_{x \rightarrow 0} \frac{\frac{\cos x - 1}{x}}{\frac{\sin x}{x}} \\ &= \frac{\lim_{x \rightarrow 0} \frac{\cos x - 1}{x}}{\lim_{x \rightarrow 0} \frac{\sin x}{x}} = \frac{0}{1} = 0 \end{aligned}$$

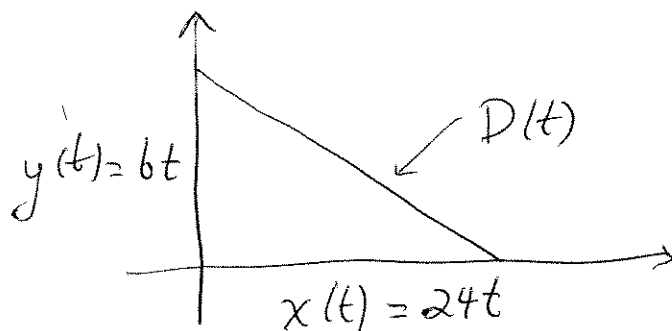
$$\begin{aligned} (b) \quad \lim_{x \rightarrow -\infty} \frac{\sqrt{9x^6 - x}}{x^3 + 1} &= \lim_{x \rightarrow \infty} \frac{\sqrt{9(-x)^6 - (-x)}}{(-x)^3 + 1} \\ &= \lim_{x \rightarrow \infty} \frac{\sqrt{9x^6 + x}}{-x^3 + 1} = \lim_{x \rightarrow \infty} \frac{\sqrt{9 + \frac{1}{x^5}}}{-1 + \frac{1}{x^3}} \\ &= \frac{\sqrt{9}}{-1} = -3 \end{aligned}$$

6. Calculate the area of the region of the plane bounded by the curves $y = x^2$ and $y = 2\sqrt{x}$.



$$\begin{aligned}
 x^2 &= 2\sqrt{x} & x^4 &= 4x & x(x^3 - 4) &= 0 \\
 \int_0^{\sqrt[3]{4}} 2\sqrt{x} - x^2 dx &= 2\left(\frac{2}{3}\right)x^{3/2} - \frac{1}{3}x^3 \Big|_0^{\sqrt[3]{4}} \\
 &= \frac{4}{3}(4^{1/3})^{3/2} - \frac{1}{3}(4^{1/3})^3 = \frac{8}{3} - \frac{4}{3} = \frac{4}{3}
 \end{aligned}$$

7. A blimp (motor-powered inflated airship) flying parallel to the ground at a speed of 24 mph releases a weather balloon that rises vertically at a rate of 6 mph. How fast is the distance between the blimp and the weather balloon increasing 5 minutes after the release?



$$5 \text{ min} = \frac{1}{12} \text{ hr.}$$

$$D\left(\frac{1}{12}\right) = \sqrt{\left(24\left(\frac{1}{12}\right)\right)^2 + \left(6\left(\frac{1}{12}\right)\right)^2} = \sqrt{4 + \frac{1}{4}} = \frac{\sqrt{17}}{2}$$

$$D(t)^2 = x(t)^2 + y(t)^2$$

$$2D(t) \frac{dD}{dt} = 2x(t) \frac{dx}{dt} + 2y(t) \frac{dy}{dt}$$

$$\frac{\sqrt{17}}{2} \frac{dD\left(\frac{1}{12}\right)}{dt} = (2)(24) + \left(\frac{1}{2}\right)(6) = 51$$

$$\frac{dD}{dt}\left(\frac{1}{12}\right) = \frac{102}{\sqrt{17}}$$

8. A function $f(x)$ is defined by

$$f(x) = \int_0^x \frac{1}{1+3t^2+2t^3} dt$$

Find the intervals on which $f(x)$ is concave upward and the intervals on which it is concave downward.

$$f'(x) = (1 + 3x^2 + 2x^3)^{-1}$$

$$f''(x) = -(1 + 3x^2 + 2x^3)^{-2} (6x + 6x^2)$$

$$= \frac{-6x(1+x)}{(1+3x^2+2x^3)^2}$$

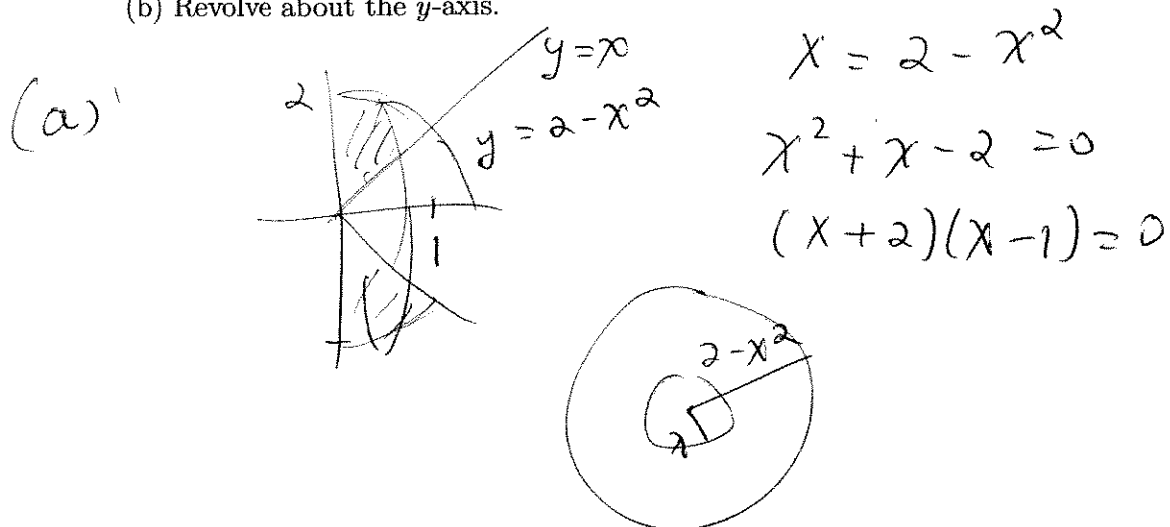
$$= 0 \text{ if } x = 0 \text{ or } -1$$

Interval	x	$1+x$	$f''(x)$	Concave
$(-\infty, -1)$	neg	neg	neg	downward
$(-1, 0)$	neg	pos	pos	upward
$(0, \infty)$	pos	pos	neg	downward

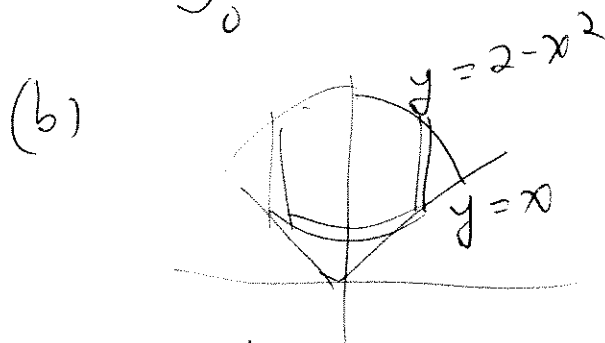
9. Set up, but do not evaluate, a definite integral for the volume of each of the solids obtained by rotating the portion of the first quadrant of the plane bounded by the curves $y = 2 - x^2$, $y = x$ and the y -axis in the following matter.

(a) Revolve about the x -axis.

(b) Revolve about the y -axis.



$$\int_0^1 \pi (2 - x^2)^2 - \pi x^2 dx$$



$$\int_0^1 2\pi x (2 - x^2 - x) dx$$

10. A function $f(x)$ has the properties $f(0) = 5$ and $f'(x) \leq -3$ for all x in $[0, 2]$. Prove that there is at least one solution to the equation $f(x) = 0$. Hint: What can you say about $f(2)$?

By the Mean Value Theorem

$$f(2) - f(0) = f'(c)(2 - 0)$$

$$f(2) - 5 = f'(c)(2) \leq -6$$

$$f(2) \leq -1 < 0$$

So $f(x) = 0$ has a solution by the Intermediate Value Theorem.