

1. Solve the differential equation

$$\frac{dy}{dx} = \csc(2x) \cot(2x)$$

with initial condition  $y(\frac{\pi}{4}) = 1$ .

$$y = \int \csc(2x) \cot(2x) dx$$

$$= -\frac{1}{2} \csc(2x) + C$$

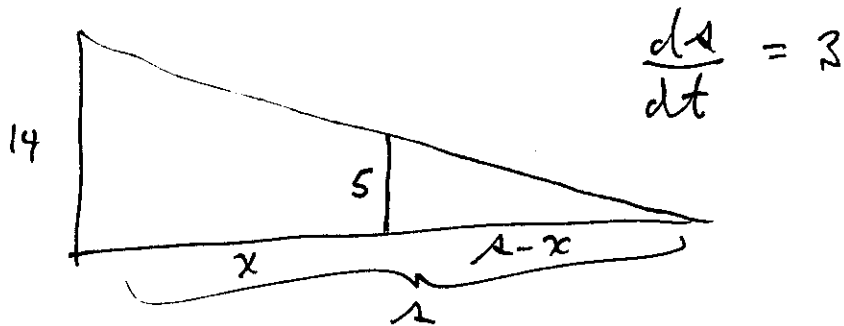
$$y\left(\frac{\pi}{4}\right) = -\frac{1}{2} \csc\left(2\frac{\pi}{4}\right) + C = 1$$

$$-\frac{1}{2 \sin\left(\frac{\pi}{2}\right)} + C = 1$$

$$-\frac{1}{2} + C = 1 \quad C = \frac{3}{2}$$

$$y = -\frac{1}{2} \csc(2x) + \frac{3}{2}$$

2. A 5-foot tall woman is walking away from a 14-foot tall lamppost. When the tip of her shadow is moving at a rate of 3 ft/sec, how fast is the woman walking?



$$\frac{s-x}{5} = \frac{s}{14}$$

$$14s - 14x = 5s$$

$$14x = 9s \quad x = \frac{9}{14}s$$

$$\frac{dx}{dt} = \frac{9}{14} \frac{ds}{dt} = \frac{9}{14} (3)$$

$$= \frac{27}{14} \text{ ft/sec.}$$

3. Find the value of  $x$  at which the function

$$f(x) = (x - x^p)^{1/q},$$

where  $p > 1$  and  $q > 1$ , attains its maximum on the interval  $[0, 1]$ .  
(You do not have to verify that it is the maximum.)

$$\begin{aligned} f'(x) &= \frac{1}{q} (x - x^p)^{\frac{1}{q}-1} (1 - px^{p-1}) \\ &= \frac{1 - px^{p-1}}{q(x - x^p)^{\frac{1}{q}-1}} = 0 \end{aligned}$$

$$1 - px^{p-1} = 0 \quad x^{p-1} = \frac{1}{p}$$

$$x = \left(\frac{1}{p}\right)^{\frac{1}{p-1}}$$

4. Let

$$f(x) = \frac{2x^2 - 1}{\sqrt{x^4 - 2}}$$

[7 points] (a) Find the horizontal asymptotes of  $f(x)$ , if it has any.

[13 points] (b) Find all the critical points of  $f(x)$ . (You do not have to determine the local maxima and minima.)

$$(a) \lim_{x \rightarrow \infty} \frac{2x^2 - 1}{\sqrt{x^4 - 2}} \cdot \frac{\frac{1}{x^2}}{\frac{1}{x^2}} = \lim_{x \rightarrow \infty} \frac{2 - \frac{1}{x^2}}{\sqrt{1 - \frac{2}{x^4}}} = 2$$

$$\lim_{x \rightarrow -\infty} \frac{2x^2 - 1}{\sqrt{x^4 - 2}} = \lim_{x \rightarrow \infty} \frac{2(-x)^2 - 1}{\sqrt{(-x)^4 - 2}} = 2$$

$$(b) f(x) = \frac{2x^2 - 1}{(x^4 - 2)^{1/2}} \quad \text{Domain: } x^4 - 2 > 0$$

$$f'(x) = \frac{4x(x^4 - 2)^{1/2} - (2x^2 - 1) \frac{1}{2} (x^4 - 2)^{-1/2} (4x^3)}{x^4 - 2} = 0$$

$$4x(x^4 - 2)^{1/2} - 2x^3(2x^2 - 1)(x^4 - 2)^{-1/2} = 0$$

$$2x(x^4 - 2) - x^3(2x^2 - 1) = 0$$

$$2x^5 - 4x - 2x^5 + x^3 = 0$$

$$x(x^2 - 4) = 0$$

$$\text{Critical points: } x = 2, x = -2$$

(note  $x = 0$  is not in the domain)

5. Prove that

$$f(x) = \frac{4}{5}x^5 - x^2 + x - 10$$

is an increasing function for all  $x$ .

To show  $g(x) = f'(x) = 4x^4 - 2x + 1 > 0$  for all  $x$

$$g'(x) = 16x^3 - 2 = 0$$

$$x^3 = \frac{1}{8} \quad x = \left(\frac{1}{8}\right)^{\frac{1}{3}} = \frac{1}{2}$$

$$g''(x) = 48x^2 \quad g''\left(\frac{1}{2}\right) > 0 \text{ so minimum}$$

$$g\left(\frac{1}{2}\right) = 4\left(\frac{1}{2}\right)^4 - 2\left(\frac{1}{2}\right) + 1 = \frac{1}{4}$$

So  $g(x) \geq \frac{1}{4}$  all  $x$