1. (a) Use differentiation formulas to calculate $f'(1)$ for

[10 points]

$$f(x) = \frac{x}{\sqrt{x^2+3}}$$

(b) Use differentiation formulas to calculate the **second** derivative of

[10 points]

$$f(x) = \sin(\sin x)$$

(a) $f(x) = \dfrac{x}{(x^2+3)^{1/2}}$

$$f'(x) = \frac{(x^2+3)^{1/2} - x\frac{1}{2}(x^2+3)^{-1/2}(2x)}{x^2+3}$$

$$f'(1) = \frac{2-\frac{1}{2}}{4} = 3/8$$

(b) $f'(x) = (\cos(\sin x))(\cos x)$

$$f''(x) = (-\sin(\sin x))(\cos x)(\cos x) + (\cos(\sin x))(-\sin x)$$

2. The slope of the tangent line to the curve

$$\sqrt{y} + axy^2 = 2$$

for $x = 0$ is 8. Find the value of a.

If $x = 0$ then $\sqrt{y} = 2$ so $y = 4$

$$\frac{1}{2} y^{-1/2} \frac{dy}{dx} + ay^2 + ax\,2y\frac{dy}{dx} = 0$$

at $x = 0$

$$\frac{1}{2}(4^{-1/2})(8) + a(4)^2 = 0$$

$$2 + 16a = 0 \qquad a = -1/8$$

3. Use the definition of the derivative and properties of limits to prove that if

$$f(x) = \sqrt{2x^2 - 1}$$

then $f'(1) = 2$.

$$f'(1) = \lim_{x \to 1} \frac{\sqrt{2x^2-1} - 1}{x-1} \cdot \frac{\sqrt{2x^2-1} + 1}{\sqrt{2x^2-1} + 1}$$

$$= \lim_{x \to 1} \frac{(2x^2-1) - 1}{(x-1)(\sqrt{2x^2-1} + 1)}$$

$$= \lim_{x \to 1} \frac{2x^2-2}{(x-1)(\sqrt{2x^2-1} + 1)}$$

$$= \lim_{x \to 1} \frac{2\cancel{(x-1)}(x+1)}{\cancel{(x-1)}(\sqrt{2x^2-1} + 1)}$$

$$= \frac{2(1+1)}{\sqrt{2-1} + 1} = \frac{4}{2} = 2$$

4. There is a number $c > 0$ such that the triangle in the first quadrant bounded by the axes and the tangent line to the curve

$$y = x^{-3}$$

at (c, c^{-3}) has area equal to 6. Find the value of c.

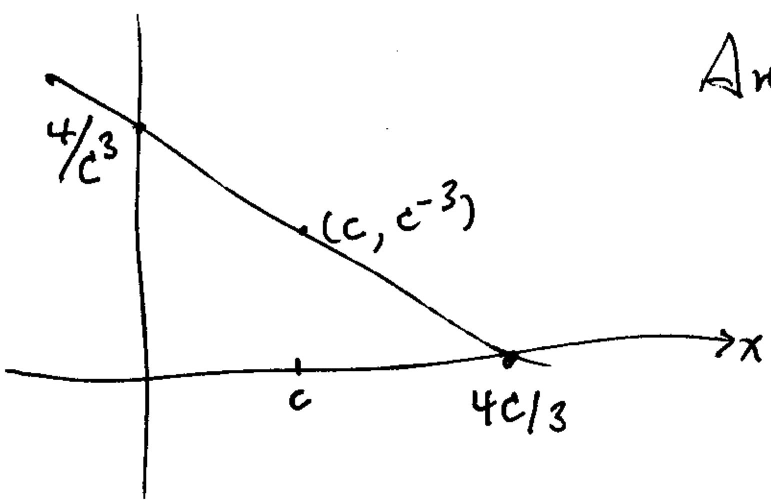
$$y' = -3y^{-4}$$

tangent line $\quad y - c^{-3} = -3c^{-4}(x - c)$

$y = 0 \quad -c^{-3} = -3c^{-4}(x - c)$

$c/3 = x - c \qquad x = 4c/3$

$x = 0 \quad y - c^{-3} = 3c^{-3} \qquad y = 4c^{-3}$



$$\text{Area} = \left(\tfrac{1}{2}\right)\left(\frac{4c}{3}\right)\left(\frac{4}{c^3}\right) = 6$$

$$\frac{8}{3c^2} = 6$$

$$c^2 = \frac{8}{18} = \frac{4}{9}$$

$$c = 2/3$$

5. Use the definition of the derivative and properties of limits to prove that if

$$f(x) = \frac{1}{g(x)}$$

when $g(x) \neq 0$, then

$$f'(x) = -\frac{g'(x)}{(g(x))^2}.$$

$$f'(x) = \lim_{h \to 0} \frac{\frac{1}{g(x+h)} - \frac{1}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{\frac{g(x) - g(x+h)}{g(x+h)\,g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{-(g(x+h) - g(x))}{h} \quad \frac{1}{g(x+h)\,g(x)}$$

$$= -\lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \lim_{h \to 0} \frac{1}{g(x+h)g(x)}$$

$$= -g'(x)\frac{1}{(g(x))^2}$$