1. Find the (absolute) maximum and minimum of

$$f(x) = \frac{\sin x + \cos x}{2}$$

on the interval $[0, \pi]$, showing your work.

$$f'(x) = \frac{1}{2}(\cos x - \sin x) = 0$$

$$\cos x = \sin x \qquad x = \pi/4$$

$$f(\pi/4) = \frac{\sin(\pi/4) + \cos(\pi/4)}{2} = \frac{\frac{\sqrt{2}}{2} + \frac{\sqrt{2}}{2}}{2} = \frac{\sqrt{2}}{2}$$

max

$$f(0) = \frac{0+1}{2} = \frac{1}{2}$$

$$f(\pi) = \frac{0+(-1)}{2} = -\frac{1}{2}$$ min

2. Use the Mean Value Theorem to prove that if $f(1) = 3$ and $f'(x) < -1$ for all $x \geq 0$, then $f(4)$ is negative.

By the MVT

$$\frac{f(4) - f(1)}{4 - 1} = f'(c) < -1$$

for some c in $[1, 4]$

$$\frac{f(4) - 3}{3} < -1$$

$$f(4) - 3 < -3$$

$$f(4) < 0$$

3. A function is defined by

$$f(x) = \frac{x}{\sqrt{2x^2+1}}.$$

[10 points] (a) Find the horizontal asymptotes of $f(x)$, if it has any.

[10 points] (b) Given that

$$f'(x) = \frac{1}{\sqrt{(2x^2+1)^3}},$$

determine the intervals on which $f(x)$ is concave up and the intervals on which $f(x)$ is concave down.

(a) $$\lim_{x\to\infty} \frac{x}{\sqrt{2x^2+1}} \frac{\frac{1}{x}}{\frac{1}{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{2+1/x^2}} = 1/\sqrt{2}$$

$$\lim_{x\to-\infty} \frac{x}{\sqrt{2x^2+1}} = \lim_{x\to\infty} \frac{-x}{\sqrt{2(-x)^2+1}}$$

$$= -\lim_{x\to\infty} \frac{x}{\sqrt{2x^2+1}} = -\frac{1}{\sqrt{2}}$$

(b) $$f'(x) = (2x^2+1)^{-3/2}$$

$$f''(x) = -\frac{3}{2}(2x^2+1)^{-5/2}(4x) = \frac{-6x}{(2x^2+1)^{5/2}}$$

Concave up on $(-\infty, 0)$

Concave down on $(0, \infty)$

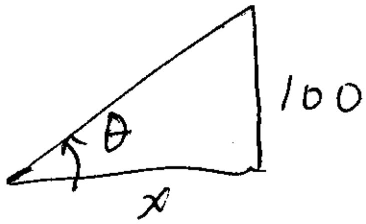
4. Given that
$$f''(x) = \sin x + 6x,$$
$f'(\pi) = 1$ and $f(\pi) = 0$, find $f(x)$.

$$f'(x) = \int \sin x + 6x \, dx$$
$$= -\cos x + 3x^2 + C$$
$$f'(\pi) = -\cos(\pi) + 3\pi^2 + C = 1$$
$$-(-1) + 3\pi^2 + C = 1 \qquad C = -3\pi^2$$

$$f(x) = \int -\cos x + 3x^2 - 3\pi^2 \, dx$$
$$= -\sin x + x^3 - 3\pi^2 x + D$$
$$f(\pi) = -\sin(\pi) + \pi^3 - 3\pi^2(\pi) + D = 0$$
$$0 + \pi^3 - 3\pi^3 + D = 0 \qquad D = 2\pi^3$$

$$f(x) = -\sin x + x^3 - 3\pi^2 x + 2\pi^3$$

5. A kite 100 feet above the ground moves horizontally at a rate of 8 feet per second. At what rate is the angle θ between the kite string and the (horizontal) ground changing when the length of the string is 200 feet? (Note: Think of the kite string as a straight line from a point on the ground to the kite.)



$$\frac{dx}{dt} = 8 \qquad \frac{d\theta}{dt} = ?$$

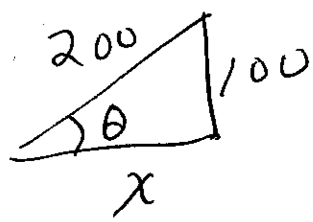
$$\tan\theta = \frac{100}{x} = 100x^{-1}$$

$$\sec^2\theta \frac{d\theta}{dt} = -100x^{-2}\frac{dx}{dt}$$

$$\sec^2\theta \frac{d\theta}{dt} = \frac{-800}{x^2}$$

$$\frac{d\theta}{dt} = -\frac{800}{x^2}\cos^2\theta$$

if



$$\cos\theta = \frac{x}{200}$$

$$\frac{d\theta}{dt} = -\frac{800}{x^2}\left(\frac{x}{200}\right)^2 = -\frac{800}{(200)^2}$$

$$\left(= -\frac{1}{50}\right)$$