1. (a) Differentiate

[10 points]

$$f(x) = \sqrt{\sin^3 x + 1}$$

(b) Calculate the **second** derivative of

[10 points]

$$f(x) = \frac{x}{x^2 + 1}$$

(a) $f(x) = (\sin^3 x + 1)^{1/2}$

$f'(x) = \frac{1}{2}(\sin^3 x + 1)^{-1/2} \frac{d}{dx}(\sin^3 x + 1)$

$= \frac{1}{2}(\sin^3 x + 1)^{-1/2}(3\sin^2 x)(\cos x)$

(b) $f'(x) = \dfrac{(x^2+1) - x(2x)}{(x^2+1)^2} = \dfrac{1-x^2}{(x^2+1)^2}$

$f''(x) = \dfrac{-2x(x^2+1)^2 - (1-x^2)2(x^2+1)(2x)}{(x^2+1)^4}$

2. Find all the values of x for which the tangent line to the curve

$$xy^2 = y + 3x^3$$

is horizontal.

$$y^2 + 2xy\underbrace{\frac{dy}{dx}}_{=0} = \underbrace{\frac{dy}{dx}}_{=0} + 9x^2$$

$$y^2 = 9x^2 \qquad y = \pm 3x$$

$$y = 3x: \quad x(3x)^2 = 3x + 3x^3 \qquad 6x^3 - 3x = 0$$

$$3x(2x^2 - 1) = 0 \quad x = 0, \; x = \pm\sqrt{\tfrac{1}{2}}$$

$$y = -3x: \quad 9x^3 = -3x + 3x^2 \qquad 6x^3 + 3x = 0$$

$$3x(2x^2 + 1) = 0 \quad x = 0$$

3. Prove that

$$\frac{\cos x}{\sin x + 2} = x$$

for some x in $[0, \pi/2]$.

Let $f(x) = x - \dfrac{\cos x}{\sin x + 2}$ which is continuous for all x since $\sin x + 2 \neq 0$

$$f(0) = 0 - \frac{1}{0+2} = -\tfrac{1}{2} < 0$$

$$f(\pi/2) = \pi/2 - \frac{0}{1+2} = \pi/2 > 0$$

so $f(x) = 0$ for some x in $[0, \pi/2]$ by IVT and for that x

$$\frac{\cos x}{\sin x + 2} = x$$

4. Find the value of a for which the limit

$$\lim_{x \to 2} \frac{x^2 - \frac{5}{3}x + a}{x - 2}$$

exists and calculate the limit.

The limit can exist only if

$$\lim_{x \to 2} x^2 - \frac{5}{3}x + a = 0$$

$$\lim_{x \to 2} x^2 - \frac{5}{3}x + a = 4 - \frac{5}{3}(2) + a = \frac{2}{3} + a$$

so $a = -2/3$

$$\lim_{x \to 2} \frac{x^2 - \frac{5}{3}x - \frac{2}{3}}{x - 2}$$

$$= \lim_{x \to 2} \frac{\cancel{(x-2)}(x + 1/3)}{\cancel{x-2}} = 2 + \frac{1}{3}$$

[2 points] 5. (a) Write the definition of the derivative $g'(x)$ of a function $g(x)$ that is the limit of a quotient in which the denominator is h.

[18 points] (b) Use the definition of part (a) to prove that if $g(x) = xf(x)$, then

$$g'(x) = xf'(x) + f(x).$$

(Do not use Leibniz' Rule.)

(a) $$g'(x) = \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

(b)
$$= \lim_{h \to 0} \frac{(x+h)f(x+h) - xf(x)}{h}$$

$$= \lim_{h \to 0} \frac{xf(x+h) + hf(x+h) - xf(x)}{h}$$

$$= \lim_{h \to 0} x\frac{f(x+h) - f(x)}{h} + \lim_{h \to 0} \frac{\cancel{h}f(x+h)}{\cancel{h}}$$

$$= x\lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + f(x)$$

$$= xf'(x) + f(x)$$