MATH 164, LECTURE 2
MIDTERM EXAMINATION
NOVEMBER 4, 2011

Name: Solutions

Instructions: Answer each question in the space provided. If the question is in several parts, carefully label the answer to each part. Do all of your work on the examination paper; scratch paper is not permitted. If you continue a problem on the back of the page, please write "continued on back".

Each problem is worth 20 points.

Problem	Score
1	
2	
3	
4	
5	
Total	

Problem 1: Consider the linear programming problem: Minimize $z = -x_1 - 2x_2$ subject to

$$x_1 + x_2 + x_3 = 2$$
$$10x_1 + x_2 + x_4 = 10$$
$$x_1, x_2, x_3, x_4 \geq 0$$

(a) Write the dual linear programming problem.
(b) Given that $(0, 2, 0, 8)^T$ is an optimal solution to the primal problem, use complementary slackness to solve the dual problem.

a) Dual problem: Maximize $w = 2y_1 + 10y_2$ Subj. to

$$y_1 + 10y_2 \leq -1$$
$$y_1 + y_2 \leq -2$$
$$y_1 \leq 0$$
$$y_2 \leq 0$$

b) Basic variables in optimal solution: x_2 and x_4.

Complementary slackness $\Rightarrow$ 2^{nd} and 4^{th} constraints for dual problem are active.

$$y_1 + y_2 = -2 \quad (x_2^* > 0)$$
$$y_2 = 0 \quad (x_4^* > 0)$$

Optimal solution for dual problem: $(-2, 0)^T$.

Problem 2: Solve the following linear programming problem using the simplex method, starting with the basis $\{x_2, x_4\}$: Minimize $z = x_1 - x_2$ subject to

$$x_1 + x_2 - x_3 = 2$$
$$3x_1 + x_2 + x_4 = 3$$
$$x_1, x_2, x_3, x_4 \geq 0$$

Dictionary:

$$X_2 = 2 + X_3 - X_1$$

$$X_4 = 3 - X_2 - 3X_1 = 3 - (2 + X_3 - X_1) - 3X_1$$

$$= 1 - 2X_1 - X_3$$

$$Z = X_1 - X_2 = X_1 - (2 + X_3 - X_1) = -2 + 2X_1 - X_3$$

Not optimal, add X_3 to basis:

$X_2 = 2 + X_3 \geq 0 \Rightarrow$ no restriction on X_3

$X_4 = 1 - X_3 \geq 0 \Rightarrow X_3 \leq 1$

Remove X_4 from basis, set $X_3 = 1$.

$$X_3 = 1 - 2X_1 - X_4$$

$$Z = -2 + 2X_1 - X_3 = -2 + 2X_1 - (1 - 2X_1 - X_4) = -3 + 4X_1 + X_4$$

Positive coefficients, so optimal solution with basis $\{X_2, X_3\}$:

Optimal solution: $(0, 3, 1, 0)^T$

Problem 3: Given a linear programming problem: Minimize $z = c^T x$ subject to $Ax = b, x \geq 0$ and its dual problem: Maximize $w = b^T y$ subject to $A^T y \leq c$, let $\overline{x} \in \mathbb{R}^n$ and $\overline{y} \in \mathbb{R}^m$ be feasible solutions to the primal and dual linear programming problems, respectively. Prove that

$$b^T \overline{y} \leq c^T \overline{x}.$$

$$b^T \overline{y} = (A\overline{x})^T \overline{y} = (\overline{x}^T A^T) \overline{y} = \overline{x}^T (A^T \overline{y}).$$

Since $\overline{x} \geq 0$ and $A^T \overline{y} \leq c$, we have

$$\overline{x}^T (A^T \overline{y}) \leq \underbrace{\overline{x}^T c}_{1\times 1} = (\overline{x}^T c)^T = c^T \overline{x}.$$

Combining these, we have

$$b^T \overline{y} = \overline{x}^T (A^T \overline{y}) \leq c^T \overline{x}$$

as desired.

Problem 4: Consider the following set of constraints:

$$2x_1 + x_2 \geq 2$$
$$\frac{1}{2}x_1 + x_2 \geq 1$$
$$x_1, x_2 \geq 0$$

(a) Show that $d = (1,4)^T$ is a direction of unboundedness for the feasible set S.
(b) Find the extreme points of S.
(c) Express $\overline{x} = (1,3)^T$ as the sum of a direction of unboundedness that is a scalar multiple of d and a convex combination of extreme points of S.

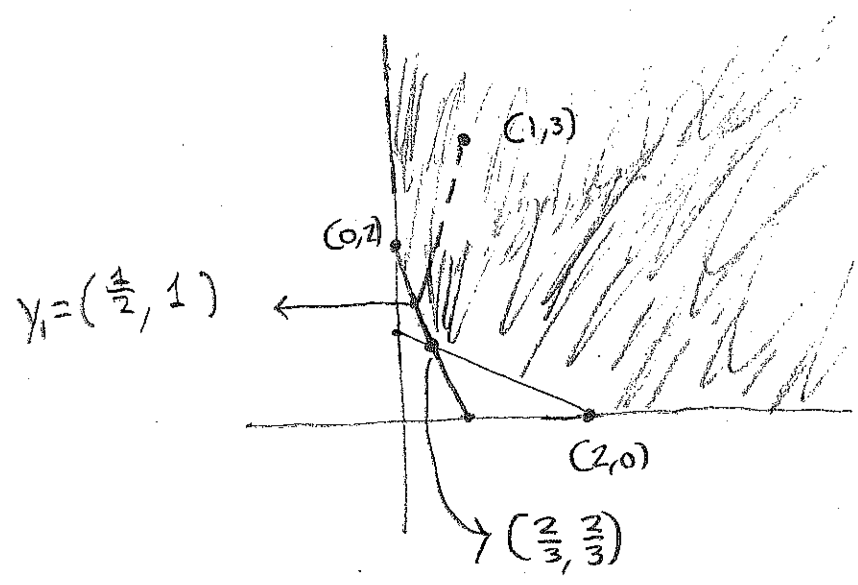
(a) We have $d \geq 0$, and $Ad = \begin{bmatrix} 2 & 1 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ \frac{9}{2} \end{bmatrix}$,

So if $x \in S$ is feasible we have $x + \gamma d \geq x \geq 0$

and $A(x+\gamma d) \geq Ax \geq b = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ for any $\gamma > 0$,

so $x + \gamma d \in S$, hence d is a direction of unboundedness.

b)



$2 - 2x_1 = 1 - \frac{1}{2}x_1$

$1 = \frac{3}{2}x_1 \Rightarrow x_1 = \frac{2}{3}, \ x_2 = \frac{2}{3}$

(continued on back)

c) Find the intersection y_1 of $\bar{X} - \gamma d$ with $2X_1 + X_2 = 2$

$$2(1-\alpha) + (3-4\alpha) = 2$$

$$5 - 6\alpha = 2 \Rightarrow \alpha = \frac{1}{2}$$

$$y_1 = (1,3) - \frac{1}{2}(1,4) = \left(\frac{1}{2}, 1\right).$$

Now express y_1 as a convex combination of $(0,2)$ and $\left(\frac{2}{3}, \frac{2}{3}\right)$:

$$\left(\frac{1}{2}, 1\right) = t \cdot (0,2) + (1-t)\left(\frac{2}{3}, \frac{2}{3}\right)$$

$$\Rightarrow (1-t) \cdot \frac{2}{3} = \frac{1}{2} \Rightarrow (1-t) = \frac{3}{4} \Rightarrow t = \frac{1}{4}.$$

We now have:

$$\bar{X} = (1,3) = \frac{1}{4} \cdot (0,2) + \frac{3}{4} \cdot \left(\frac{2}{3}, \frac{2}{3}\right) + \overset{\frac{1}{2} \cdot d}{\overset{\parallel}{\left(\frac{1}{2}, 2\right)}}$$

Problem 5: Suppose that the feasible set S for linear programming problem: Minimize $z = c^T x$ subject to $Ax = b$, $x \geq 0$, has a direction of unboundedness $d \neq 0$. Prove that if $c^T d < 0$ then the linear programming problem is unbounded, i.e. has no optimal solution.

Fix $x \in S$. Then for $\gamma > 0$ we have

$x + \gamma d \in S$ and $z(x+\gamma d) = c^T(x+\gamma d) = c^T x + \gamma c^T d$.

Since $c^T d < 0$, we have

$$\lim_{\gamma \to \infty} z(x+\gamma d) = \lim_{\gamma \to \infty} c^T x + \gamma \cdot c^T d = -\infty,$$

So the problem is unbounded.