Name: Solutions

Instructions: Answer each question in the space provided. If the question is in several parts, carefully label the answer to each part. Do all of your work on the examination paper; scratch paper is not permitted. If you continue a problem on the back of the page, please write "continued on back".

Each problem is worth 20 points.

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Problem 1: Consider the linear programming problem: Minimize \( z = -x_1 - 2x_2 \) subject to
\[
\begin{align*}
x_1 + x_2 + x_3 &= 2 \\
10x_1 + x_2 + x_4 &= 10 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{align*}
\]
(a) Write the dual linear programming problem.
(b) Given that \((0, 2, 0, 8)^T\) is an optimal solution to the primal problem, use complementary slackness to solve the dual problem.

\begin{align*}
a) \text{Dual problem: } & \max w = 2y_1 + 10y_2 \text{ subj. to } \\
y_1 + 10y_2 &\leq -1 \\
y_1 + y_2 &\leq -2 \\
y_1 &\leq 0 \\
y_2 &\leq 0
\end{align*}

b) Basic variables in optimal solution: \(x_2\) and \(x_4\).

Complementary slackness \(\Rightarrow\) 2nd and 4th constraints for dual problem are active.

\[
\begin{align*}
y_1 + y_2 &= -2 \quad (x_2^x > 0) \\
y_2 &= 0 \quad (x_4^x > 0)
\end{align*}
\]

Optimal solution for dual problem: \((-2, 0)^T\).
**Problem 2:** Solve the following linear programming problem using the simplex method, starting with the basis \( \{x_2, x_4\} \): Minimize \( z = x_1 - x_2 \) subject to
\[
\begin{align*}
x_1 + x_2 - x_3 &= 2 \\
3x_1 + x_2 + x_4 &= 3 \\
x_1, x_2, x_3, x_4 &\geq 0
\end{align*}
\]

**Dictionary:**
\[
\begin{align*}
x_2 &= 2 + x_3 - x_1 \\
x_4 &= 3 - x_2 - 3x_1 = 3 - (2 + x_3 - x_1) - 3x_1 = 1 - 2x_1 - x_3
\end{align*}
\]
\[
z = x_1 - x_2 = x_1 - (2 + x_3 - x_1) = -2 + 2x_1 + x_3
\]

Not optimal, add \( x_3 \) to basis:
\[
x_2 = 2 + x_3 \geq 0 \Rightarrow \text{no restriction on } x_3 \\
x_4 = 1 - x_3 \geq 0 \Rightarrow x_3 \leq 1
\]

Remove \( x_4 \) from basis, set \( x_3 = 1 \).
\[
x_3 = 1 - 2x_1 - x_4
\]
\[
z = -2 + 2x_1 - x_3 = -2 + 2x_1 - (1 - 2x_1 - x_4) = -3 + 4x_1 + x_4
\]

Positive coefficients, so optimal solution with basis \( \{x_2, x_3\} \):

Optimal solution: \((0, 3, 1, 0)^T\)
Problem 3: Given a linear programming problem: Minimize \( z = c^T x \) subject to \( Ax = b, x \geq 0 \) and its dual problem: Maximize \( w = b^T y \) subject to \( A^T y \leq c \), let \( \bar{x} \in \mathbb{R}^n \) and \( \bar{y} \in \mathbb{R}^m \) be feasible solutions to the primal and dual linear programming problems, respectively. Prove that
\[
b^T \bar{y} \leq c^T \bar{x}.
\]

\[
b^T \bar{y} = (Ax)^T \bar{y} = (\bar{x}^T A^T) \bar{y} = \bar{x}^T (A^T \bar{y}).
\]

Since \( \bar{x} \geq 0 \) and \( \bar{A}^T \bar{y} \leq \bar{c} \), we have
\[
\bar{x}^T (A^T \bar{y}) \leq \bar{x}^T \bar{c} = (\bar{x}^T \bar{c})^T_{1 \times 1} = \bar{c}^T \bar{x}.
\]

Combining these, we have
\[
b^T \bar{y} = \bar{x}^T (A^T \bar{y}) \leq \bar{c}^T \bar{x}
\]

as desired.
Problem 4: Consider the following set of constraints:

\[
\begin{align*}
2x_1 + x_2 & \geq 2 \\
\frac{1}{2}x_1 + x_2 & \geq 1 \\
x_1, x_2 & \geq 0
\end{align*}
\]

(a) Show that \(d = (1, 4)^T\) is a direction of unboundedness for the feasible set \(S\).
(b) Find the extreme points of \(S\).
(c) Express \(\bar{x} = (1, 3)^T\) as the sum of a direction of unboundedness that is a scalar multiple of \(d\) and a convex combination of extreme points of \(S\).

\[\text{(a)} \quad \text{We have } d \geq 0, \text{ and } A d = \begin{bmatrix} 2 & 1 \\ \frac{1}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 4 \end{bmatrix} = \begin{bmatrix} 6 \\ \frac{9}{2} \end{bmatrix}.\]

So if \(x \in S\) is feasible we have \(x + \gamma d \geq x \geq 0\).

\[\quad \text{and } A (x + \gamma d) \geq A x \geq b = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \text{ for any } \gamma > 0.\]

So \(x + \gamma d \in S\), hence \(d\) is a direction of unboundedness.

\[\text{(b)} \quad \gamma_1 = (\frac{3}{2}, 1)\]

\[
\begin{align*}
2 - 2x_1 &= 1 - \frac{1}{2}x_1 \\
1 &= \frac{3}{2}x_1 \Rightarrow x_1 = \frac{2}{3}, \quad x_2 = \frac{2}{3}
\end{align*}
\]

(Continued on back)
C) Find the intersection \( y_1 \) of \( X - Y_d \) with \( 2X_1 + X_2 = 2 \)

\[
2(1-\alpha) + (3-4\alpha) = 2 \\
5 - 6\alpha = 2 \implies \alpha = \frac{1}{2}
\]

\[y_1 = (1, 3) - \frac{1}{2} (1, 4) = \left( \frac{1}{2}, 1 \right).\]

Now express \( y_1 \) as a convex combination of \( (0, 2) \) and \( \left( \frac{2}{3}, \frac{2}{3} \right) \):

\[
\left( \frac{1}{2}, 1 \right) = t \cdot (0, 2) + (1-t) \left( \frac{2}{3}, \frac{2}{3} \right)
\]

\[
\implies (1-t) \cdot \frac{2}{3} = \frac{1}{2} \implies (1-t) = \frac{3}{4} \implies t = \frac{1}{4}.
\]

We now have:

\[X_1 = \left( \frac{1}{3} \right) = \frac{1}{4} \cdot (0, 2) + \frac{3}{4} \cdot \left( \frac{2}{3}, \frac{2}{3} \right) + \left( \frac{1}{2}, 2 \right)\]
Problem 5: Suppose that the feasible set $S$ for linear programming problem: Minimize $z = c^T x$
subject to $Ax = b$, $x \geq 0$, has a direction of unboundedness $d \neq 0$. Prove that if $c^T d < 0$ then the
linear programming problem is unbounded, i.e. has no optimal solution.

Fix $x \in S$. Then for $\gamma > 0$ we have

$$x + \gamma d \in S \text{ and } z(x + \gamma d) = c^T (x + \gamma d) = c^T x + \gamma c^T d.$$ 

Since $c^T d < 0$, we have

$$\lim_{\gamma \to \infty} z(x + \gamma d) = \lim_{\gamma \to \infty} c^T x + \gamma c^T d = -\infty,$$

So the problem is unbounded.