MATH 164, LECTURE 2
FINAL EXAMINATION
DECEMBER 8, 2011

Name: Solutions

Instructions: Answer each question in the space provided. If the question is in several parts, carefully label the answer to each part. Do all of your work on the examination paper; scratch paper is not permitted. If you continue a problem on the back of the page, please write "continued on back".

Each problem is worth 20 points.

Problem	Score
1	
2	
3	
4	
5	
6	
7	
8	
9	
10	
Total	

Problem 1: Use Newton's method to find a stationary point of

$$f(x_1, x_2) = x_1^2 x_2 - 2x_1 x_2 + x_2^2 - 1,$$

starting with $\bar{x}_0 = (1,1)^T$.

$$\nabla f(x) = \begin{bmatrix} 2x_1x_2 - 2x_2 \\ x_1^2 - 2x_1 + 2x_2 \end{bmatrix}$$

$$\nabla^2 f(x) = \begin{bmatrix} 2x_2 & 2x_1 - 2 \\ 2x_1 - 2 & 2 \end{bmatrix}$$

$$\nabla f(1,1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$\nabla^2 f(1,1) = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow \left(\nabla^2 f(1,1)\right)^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}$$

$$\bar{X}_1 = \bar{X}_0 - \left(\nabla^2 f(\bar{X}_0)\right)^{-1} \cdot \nabla f(\bar{X}_0) = \begin{bmatrix} 1 \\ 1 \end{bmatrix} - \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$

$$\nabla f(\bar{X}_1) = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Rightarrow \bar{X}_1 = \begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}$$ is a stationary pt.

Problem 2: Consider the linear programming problem: Minimize $5x_1 + 4x_2$ subject to

$$x_1 + 2x_2 - x_3 = 2$$
$$2x_1 + x_2 = 3$$
$$x_1, x_2, x_3 \geq 0$$

(a) Write the dual linear programming problem. (b) Given that the optimal solution to the dual problem is $(1, 2)^T$, use complementary slackness to solve the given problem.

(a) Maximize $W = 2y_1 + 3y_2$ subject to

$$y_1 + 2y_2 \leq 5$$
$$2y_1 + y_2 \leq 4$$
$$-y_1 \leq 0$$

(b) Check for active constraints in dual:

$1 + 2\cdot(2) = 5$ (active)

$2\cdot(1) + 2 = 4$ (active)

$-(1) < 0$ (inactive) $\Rightarrow X_3 = 0.$

$X_3 = 0$, now solve

$$X_1 + 2X_2 = 2$$
$$2X_1 + X_2 = 3$$
$\Rightarrow X_1 = \frac{4}{3}, X_2 = \frac{1}{3}$

So optimal solution is $(4/3, 1/3, 0)^T.$

Problem 3: Find the local minimizers of $f(x) = x_1^3 + 2x_1x_2 - x_1 + 2x_2$ subject to

$$x_1 + x_2 \geq 2$$
$$x_2 \geq 0$$

(a) if the first constraint is active and the second is not (b) if the second constraint is active and the first is not.

$$\nabla f(x) = \begin{bmatrix} 3x_1^2 + 2x_2 - 1 \\ 2x_1 + 2 \end{bmatrix}$$

$$A^T\lambda = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 \\ \lambda_1 + \lambda_2 \end{bmatrix}$$

(a) Complementary slackness $\Rightarrow \lambda_2 = 0$,

$$3x_1^2 + 2x_2 - 1 = \lambda_1$$
$$2x_1 + 2 = \lambda_1$$
$$x_1 + x_2 = 2 \Rightarrow x_2 = 2 - x_1$$

$$\Rightarrow 3x_1^2 + 2(2 - x_1) - 1 = \lambda_1 = 2x_1 + 2$$

$$3x_1^2 - 4x_1 + 1 = 0, \quad x_1 = \frac{4 \pm \sqrt{16-12}}{6} = 1, \tfrac{1}{3}$$

Stationary pts: $(1, 1, 4, 0)$, $(\tfrac{1}{3}, \tfrac{5}{3}, \tfrac{8}{3}, 0)$

$$\nabla^2 f(x) = \begin{bmatrix} 6x_1 & 2 \\ 2 & 0 \end{bmatrix}, \quad \hat{A}_+ = [1 \; 1] \Rightarrow Z_+ = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

(continued on back)

$$Z_+^T \nabla^2 f(x) Z_+ = [1\ -1] \begin{bmatrix} 6x_1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} = [1\ -1] \begin{bmatrix} 6x_1 - 2 \\ -2 \end{bmatrix} = 6x_1$$

pos. def. $\Leftrightarrow x_1 > 0$, so

$(1,1,4,0)$ and $(1/3, 5/3, 8/3, 0)$ are local mins.

b) $\lambda_1 = 0,\ x_2 = 0.$

$$\begin{aligned} 3x_1^2 - 1 &= 0 \\ 2x_1 + 2 &= \lambda_2 \end{aligned} \Rightarrow x_1 = \pm \frac{\sqrt{3}}{3}$$

Stationary pts: $\left(\frac{\sqrt{3}}{3}, 0, 0, 2 + \frac{2\sqrt{3}}{3}\right)$, $\left(\frac{-\sqrt{3}}{3}, 0, 0, 2 - \frac{2\sqrt{3}}{3}\right)$

$$\hat{A}_+ = [0\ 1] \Rightarrow Z_+ = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$Z_+^T \nabla^2 f(x) Z_+ = [1\ 0] \begin{bmatrix} 6x_1 & 2 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = 6x_1$$

pos. def. $\Leftrightarrow x_1 > 0$, so

$\left(\frac{\sqrt{3}}{3}, 0, 0, 2 + \frac{2\sqrt{3}}{3}\right)$ is a local min.

$\left(\frac{\sqrt{3}}{3}, 0, 0, 2 - \frac{2\sqrt{3}}{3}\right)$ is <u>not</u> a local min.

Problem 4: Find all of the local minimizers for the unconstrained problem: Minimize

$$f(x) = f(x_1, x_2) = x_1^3 - x_2^3 + x_1^2 - x_1 + 3x_2.$$

$$\nabla f(x) = \begin{bmatrix} 3x_1^2 + 2x_1 - 1 \\ -3x_2^2 + 3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow x_2 = \pm 1, \quad x_1 = \frac{-2 \pm \sqrt{4+12}}{6} = -1, \ 1/3$$

Stationary pts: $(-1, 1), (-1, -1), (1/3, 1), (1/3, -1)$

$$\nabla^2 f(x) = \begin{bmatrix} 6x_1 + 2 & 0 \\ 0 & -6x_2 \end{bmatrix}$$

Positive-definite $\Leftrightarrow$ $x_2 < 0$, $6x_1 + 2 > 0 \Leftrightarrow x_1 > -1/3$

so $(1/3, -1)$ is a local min., the rest

are <u>not</u>.

Problem 5: Find the local minimizers of $f(x) = x_1^2 + x_2^2 - x_2 x_3$ subject to

$$x_1 - x_2 - x_3 = 2$$

$$\nabla f(x) = \begin{bmatrix} 2x_1 \\ 2x_2 - x_3 \\ -x_2 \end{bmatrix}$$

$$A[u\ v\ w] = u - v - w = 0 \Leftrightarrow w = u - v$$

$$\Rightarrow \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} u + \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} w \Rightarrow Z = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & -1 \end{bmatrix}$$

$$Z^T \nabla f(x) = \begin{bmatrix} 2x_1 - x_2 \\ 3x_2 - x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$\begin{aligned} 2x_1 - x_2 &= 0 \\ 3x_2 - x_3 &= 0 \\ x_1 - x_2 - x_3 &= 2 \end{aligned} \Rightarrow \begin{aligned} & x_2 = 2x_1 \\ & x_3 = 3x_2 = 6x_1 \\ & -7x_1 = 2 \Rightarrow x_1 = \frac{-2}{7} \end{aligned}$$

Stationary pts: $\left(-\frac{2}{7}, \frac{-4}{7}, \frac{-12}{7}\right)$

$$\nabla^2 f(x) = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 2 & -1 \\ 0 & -1 & 0 \end{bmatrix} \Rightarrow Z^T \nabla^2 f(x) Z$$

$$= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ -1 & 3 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 4 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -1 \\ 0 & 7/2 \end{bmatrix}$$

is pos. def, so $(-2/7, -4/7, -12/7)$ is the unique local min.

Problem 6: Consider the problem: Minimize $z = 2x_1 - x_2 + 3x_3$ subject to

$$x_1 - x_2 + 2x_3 \geq 2$$
$$x_1 + 2x_2 - x_3 \geq 0$$
$$2x_1 + x_3 \geq 0$$

(a) Show that $x = (0,1,2)^T$ is a feasible solution and identify which constraints are active and which are inactive at that point.
(b) Show that $p = (1,0,-3)^T$ is a feasible direction at $x = (0,1,2)^T$.
(c) Determine the largest value of $\alpha > 0$ such that $x + \alpha p$ remains feasible, using x and p as in part (b).
(d) Find all the feasible directions $p = (p_1, p_2, p_3)^T$ at $x = (0,1,2)^T$.

(a)

$0 - 1 + 2 \cdot 2 = 3 > 2$ (inactive)

$0 + 2 - 2 = 0 = 0$ (active)

$2 \cdot (0) + 2 = 2 > 0$ (inactive)

(b) Just need to check active constraints:

$a_2^T p = p_1 + 2p_2 - p_3 = 1 + 2 \cdot 0 + 3 = 4 > 0$

So p is a feasible direction at X.

(c) $a_1^T \cdot p = 1 - 0 + 2(-3) = -5$

$a_3^T \cdot p = 2 \cdot 1 + (-3) = -1$

$\bar{\alpha} = \min \left\{ \frac{3-2}{5}, \frac{2-0}{1} \right\} = \frac{1}{5}$

(d) Just need $a_2^T p = p_1 + 2p_2 - p_3 \geq 0.$

Problem 7: Find all local minimizers of $f(x) = x_1^2 + x_2$ subject to

$$x_1^2 + 2x_2 = 2$$
$$-2x_1 + x_2 + x_3 = 1$$

$\Rightarrow$ $g_1(x) = X_1^2 + 2X_2 - 2 = 0$
$g_2(x) = -2X_1 + X_2 + X_3 - 1 = 0$

$$\nabla g(x)^T = \begin{bmatrix} 2X_1 & 2 & 0 \\ -2 & 1 & 1 \end{bmatrix} \Rightarrow \text{rank}(\nabla g(x)^T) = 2 \text{ for all } X,$$

So all feasible pts. are regular.

$$\mathcal{L}(X, \lambda) = X_1^2 + X_2 - \lambda_1(X_1^2 + 2X_2 - 2) - \lambda_2(-2X_1 + X_2 + X_3 - 1)$$

$$\Rightarrow \nabla_x \mathcal{L}(X,\lambda) = \begin{bmatrix} 2X_1 - 2\lambda_1 X_1 + 2\lambda_2 \\ 1 - 2\lambda_1 - \lambda_2 \\ -\lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\Rightarrow \lambda_2 = 0, \quad \lambda_1 = 1/2, \quad X_1 = 0$$
$$2X_2 = 2 \Rightarrow X_2 = 1$$
$$1 + X_3 = 1 \Rightarrow X_3 = 0$$

Stationary pt: $(0, 1, 0, 1/2, 0)$

$$\nabla_{xx} \mathcal{L}(X, \lambda) = \begin{bmatrix} 2 - 2\lambda_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

(continued on back)

$$\Rightarrow \nabla_{xx} \mathcal{L}(0,1,0,\tfrac{1}{2},0) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\nabla g(0,1,0)^T = \begin{bmatrix} 0 & 2 & 0 \\ -2 & 1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 & 0 \\ -2 & 1 & 1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 2v \\ -2u + v + w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow v = 0, w = 2u$$

$$Z = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$$

$$Z^T \nabla_{xx} \mathcal{L}(0,1,0,\tfrac{1}{2},0) Z = \begin{bmatrix} 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 1 > 0$$

pos. def. $\Rightarrow (0,1,0,\tfrac{1}{2},0)$ is the unique local minimizer.

Problem 8:

(a) Let A be a $m \times n$ matrix, and let $b \in \mathbb{R}^m$. Prove that the set

$$S = \{x \in \mathbb{R}^n : Ax \geq b\}$$

is convex.

For parts (b) and (c), sketch the feasible set determined by the given constraints, and state whether this set is convex (you do not need to prove convexity).

(b)

$$\begin{aligned} x_1^2 - x_2 &\geq 0 \\ x_1 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(c)

$$\begin{aligned} x_1^2 + x_2^2 &\leq 1 \\ x_1, x_2 &\geq 0 \end{aligned}$$

a) Let $x, y \in S$ and $0 \leq t \leq 1$.

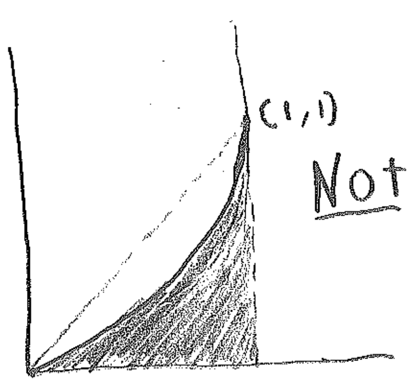
Then

$$\begin{aligned} A(t \cdot x + (1-t)y) &= t \cdot Ax + (1-t) \cdot Ay \\ &\geq t \cdot b + (1-t) \cdot b = b \end{aligned}$$

Since $Ax \geq b$, $Ay \geq b$ and $t \geq 0$, $(1-t) \geq 0$.

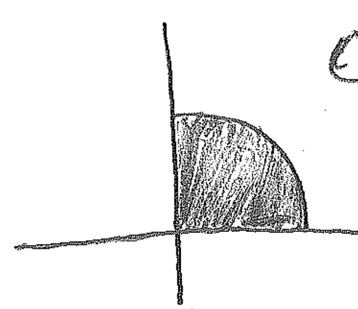
So $t \cdot x + (1-t) \cdot y \in S$, hence S is convex

b)



Not convex

c)



Convex

Problem 9: Consider the problem: Minimize $f(x) = x_1 - 2x_2$ subject to

$$x_1 + 2x_2 \geq 1$$
$$x_1 \geq x_2^2$$

Show that $(1,1)^T$ is a local minimizer by showing that $(1,1)^T$ satisfies the sufficient conditions presented in the course.

$$\nabla g(x)^T = \begin{bmatrix} 1 & 2 \\ 1 & -2x_2 \end{bmatrix} \Rightarrow \nabla g(1,1)^T = \begin{bmatrix} 1 & 2 \\ 1 & -2 \end{bmatrix},$$

$\text{rank}(\nabla g(1,1)^T) = 2$, so $(1,1)^T$ is regular.

$$\mathcal{L}(x,\lambda) = x_1 - 2x_2 - \lambda_1(x_1 + 2x_2 - 1) - \lambda_2(x_1 - x_2^2)$$

$$\nabla_x \mathcal{L}(x,\lambda) = \begin{bmatrix} 1 - \lambda_1 - \lambda_2 \\ -2 - 2\lambda_1 + 2\lambda_2 x_2 \end{bmatrix}$$

$g_1(1,1) = 1 + 2 - 1 = 2 > 0$ (inactive) $\Rightarrow \lambda_1 = 0$

$g_2(1,1) = 1 - 1^2 = 0$ (active)

$$\nabla_x \mathcal{L}(1,1,0,\lambda_2) = \begin{bmatrix} 1 - \lambda_2 \\ -2 + 2\lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow \lambda_2 = 1.$$

So $(1,1,0,1)$ is stationary.

(continued on back).

Problem 10:

(a) Prove that if $y \in \mathbb{R}^n$ and $y^T y = 0$, then $y = 0$.

(b) Let Z be a $m \times n$ matrix, prove that if $Z^T Z v = 0$ for some $v \in \mathbb{R}^n$, then $Zv = 0$. (Hint: Consider $v^T Z^T Z v$).

(c) Now consider the problem: Minimize $f(x) = f(x_1, \ldots, x_n)$ subject to $Ax = b$, where A is a $m \times n$ matrix of rank m. Let Z be a basis null-space matrix for A, and recall the decomposition

$$\nabla f(x_*) = Z v_* + A^T \lambda_*$$

for some $v_* \in \mathbb{R}^{n-m}$, $\lambda_* \in \mathbb{R}^m$. Prove that if $Z^T \nabla f(x_*) = 0$ then $\nabla f(x_*) = A^T \lambda_*$.

a) $y = (y_1, \ldots, y_n)$, $y^T y = y_1^2 + \cdots + y_n^2 = 0$

then $y_1^2 = \cdots = y_n^2 = 0$ so $y_1 = \cdots = y_n = 0$, $y = 0$.

b) $Zv \in \mathbb{R}^n$, $(Zv)^T Zv = v^T Z^T Z v$

$= v^T (Z^T Z v) = 0$

so $Zv = 0$ by (a)

c) $0 = Z^T \nabla f(x_*) = Z^T Z v_* + Z^T A^T \lambda_*$.

Since Z is a basis nullspace matrix for A,

$AZ = 0 \Rightarrow (AZ)^T = Z^T A^T = 0$.

So $Z^T Z v_* = 0 \Rightarrow Z v_* = 0$ by (b),

we then have $\nabla f(x_*) = \cancel{Z v_*} + A^T \lambda_* = A^T \lambda_*$.