1. Consider the following set of constraints:

$$\begin{aligned} 2x_1 - x_2 + x_3 &\geq 2 \\ 2x_1 + 3x_2 &\geq 2 \\ x_1 - 2x_2 + 3x_3 &\geq 1 \end{aligned}$$

(a) Show that $\bar{x} = (1,0,1)^T$ is feasible and indicate which constraints are active and which are inactive. (b) Show that $p = (-1,2,-2)^T$ is a feasible direction at $\bar{x}$. (c) Find the maximum value $\bar{\alpha}$ among all $\alpha > 0$ such that $\bar{x} + \alpha p$ is feasible.

[7 pts] (a) $2 - 0 + 1 = 3 > 2$ inactive

$2 + 0 = 2 = 2$ active

$1 - 0 + 3 = 4 > 1$ inactive

[5 pts] (b) $\begin{bmatrix} 2 & 3 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -2 \end{bmatrix} = -2 + 6 + 0 = 4 > 0$

[10 pts] (c) $\begin{bmatrix} 2 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1-\alpha \\ 0+2\alpha \\ 1-2\alpha \end{bmatrix} = 2 - 2\alpha - 2\alpha + 1 - 2\alpha$

$= 3 - 6\alpha \geq 2 \quad \alpha \leq 1/6$

$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 1-\alpha \\ 0+2\alpha \\ 1-2\alpha \end{bmatrix} = 1 - \alpha - 4\alpha + 3 - 6\alpha$

$= 4 - 11\alpha \geq 1 \quad \alpha \leq 3/11$

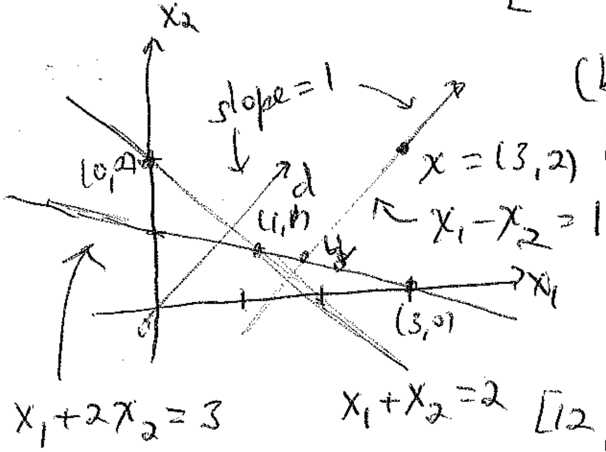
$\bar{\alpha} = 1/6$

2. Let $S$ be the feasible set defined by the constraints

$$\begin{aligned} x_1 + 2x_2 &\geq 3 \\ x_1 + x_2 &\geq 2 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(a) Show that $d = (2,2)^T$ is a direction of unboundedness for $S$. (b) Find the extreme points of $S$. (c) Write $x = (3,2)^T$ as the sum of a direction of unboundedness that is a scalar multiple of $d$ and a convex combination of extreme points of $S$.

(a) [3 pts] $d \geq 0 \quad Ad = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ 4 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$




(b) $(3,0), (0,2)$ [5 pts]

$x_1 + 2x_2 = 3$

$x_1 + x_2 = 2$

$x_2 = 1, x_1 = 1$ so $(1,1)$

(c) [12 pts]

$x_1 + 2x_2 = 3$

$x_1 - x_2 = 1$

$3x_2 = 2 \quad x_2 = 2/3 \quad x_1 = 5/3 \quad y = (5/3, 2/3)$

$$\begin{bmatrix} 5/3 \\ 2/3 \end{bmatrix} = \alpha \begin{bmatrix} 1 \\ 1 \end{bmatrix} + (1-\alpha)\begin{bmatrix} 3 \\ 0 \end{bmatrix} \qquad \alpha = 2/3$$

$3 = 2\gamma + 5/3 \quad \gamma = 2/3$

$$x = \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \gamma \begin{bmatrix} 2 \\ 2 \end{bmatrix} + \begin{bmatrix} 5/3 \\ 2/3 \end{bmatrix}$$

$$x = \begin{bmatrix} 4/3 \\ 4/3 \end{bmatrix} + \frac{2}{3}\begin{bmatrix} 1 \\ 1 \end{bmatrix} + \frac{1}{3}\begin{bmatrix} 3 \\ 0 \end{bmatrix}$$

3. Let $S$ be the feasible set for the constraints $Ax = b, x \geq 0$, let $d$ be a direction of unboundedness for $S$ and let $v_1, \ldots, v_k$ be the extreme points of $S$. Prove that if

$$x_\gamma = \gamma d + \sum_{i=1}^{k} \alpha_i v_i$$

where $\alpha_i \geq 0$ and $\sum_{i=1}^{k} \alpha_i = 1$, then $x_\gamma \in S$ for all $\gamma > 0$.

$d$ a direction of unboundedness then $d \geq 0$ and $Ad = 0$
$v_i \in S$ so $v_i \geq 0$ and $Av_i = b$. Since $\alpha_i \geq 0$ and
$\gamma > 0$ then $x_\gamma \geq 0$

$$\begin{aligned} Ax_\gamma &= A(\gamma d + \sum_{i=1}^{k} \alpha_i v_i) \\ &= \gamma A d + \sum_{i=1}^{k} \alpha_i A v_i \\ &= \gamma (0) + \sum_{i=1}^{k} \alpha_i (b) \\ &= 0 + b \sum_{i=1}^{k} \alpha_i = b \end{aligned}$$

4. Given the linear programming problem below, do the following: (a) Show that $\{x_2, x_3\}$ is a basis. (b) Show that the corresponding basic solution is feasible. (c) Write the dictionary for the basis $\{x_2, x_3\}$. (d) Write the objective function $z$ in terms of the nonbasic variables.

Minimize $z = x_1 + 2x_2 + x_3 - x_4$

subject to

$$\begin{aligned} -x_1 + 2x_2 \quad\quad\;\; + x_4 &= 3 \\ 2x_1 - x_2 + 2x_3 + x_4 &= 2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(a) [2 pts]
$$\begin{array}{c} \quad\;\; x_2 \quad x_3 \\ \det \begin{bmatrix} 2 & 0 \\ -1 & 2 \end{bmatrix} = 4 \neq 0 \end{array}$$

(b) [4 pts] $x_1 = x_4 = 0 \quad x_2 = 3/2 \quad x_3 = 7/4$

$x = (0, 3/2, 7/4, 0) \geq 0$

(c) [12 pts] $2x_2 = 3 + x_1 - x_4 \qquad x_2 = 3/2 + 1/2\, x_1 - 1/2\, x_4$

$2x_1 - (3/2 + 1/2\, x_1 - 1/2\, x_4) + 2x_3 + x_4 = 2$

$3/2\, x_1 + 3/2\, x_4 + 2x_3 = 7/2$

$x_2 = 3/2 + 1/2\, x_1 - 1/2\, x_4$

$x_3 = 7/4 - 3/4\, x_1 - 3/4\, x_4$

(d) [2 pts] $z = x_1 + 2(3/2 + 1/2\, x_1 - 1/2\, x_4)$
$+ (7/4 - 3/4\, x_1 - 3/4\, x_4) - x_4$

$(= 19/4 + 5/4\, x_1 - 11/4\, x_4)$

5. Prove that the set

$$S = \{x = (x_1, \ldots, x_n) \in \mathbf{R}^n : x_1 \geq \ldots \geq x_j \geq x_{j+1} \geq \ldots \geq x_n \geq 0\}$$

is convex.

Let $x = (x_1, \ldots, x_n)$, $y = (y_1, \ldots, y_n) \in S$

and let $z = \alpha x + (1-\alpha) y$, $0 \leq \alpha \leq 1$

Then $z = (z_1, \ldots, z_n)$

$= (\alpha x_1 + (1-\alpha) y_1, \ldots, \alpha x_n + (1-\alpha) y_n)$

$z_n \geq 0$ since $x_n, y_n, \alpha, 1-\alpha \geq 0$

$x_j \geq x_{j+1}$ and $\alpha \geq 0$ implies $\alpha x_j \geq \alpha x_{j+1}$

$y_j \geq y_{j+1}$ and $(1-\alpha) \geq 0$ implies

$(1-\alpha) y_j \geq (1-\alpha) y_{j+1}$

so $z_j \geq z_{j+1}$ and $z \in S$