1. Consider the linear programming problem

Minimize $4x_1 + x_2 + 2x_3$
subject to

$$\begin{aligned} x_1 - x_2 + x_3 &\geq 2 \\ 2x_1 + x_2 &\geq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(a) Write the dual linear programming problem. (b) Given that the optimal solution to the dual problem is $(2, 1)$, use complementary slackness to solve the given problem.

[7 points] (a) Maximize $2y_1 + y_2$
subject to
$$\begin{aligned} y_1 + 2y_2 &\leq 4 \\ -y_1 + y_2 &\leq 1 \\ y_1 &\leq 2 \\ y_1, y_2 &\geq 0 \end{aligned}$$

[13 points] (b) $2(2) + (1) = 5$

so
$$\begin{aligned} 4x_1 + x_2 + 2x_3 &= 5 \\ x_1 - x_2 + x_3 &= 2 \\ 2x_1 + x_2 &= 1 \end{aligned}$$

$x_1 = 1/2$, $x_2 = 0$, $x_3 = 3/2$

2. Determine the values of a such that

$$f(x) = f(x_1, x_2) = x_1^2 + ax_2^2 - 4x_1x_2 - x_1 + 1$$

has a local minimizer and find the local minimizers.

$$\nabla f(x) = \begin{bmatrix} 2x_1 - 4x_2 - 1 \\ 2ax_2 - 4x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

No solution if $a = 4$; $2ax_2 = 4x_1$, $x_1 = \frac{a}{2}x_2$

$2\left(\frac{a}{2}x_2\right) - 4x_2 = 1$, $x_2 = \frac{1}{a-4}$, $x_1 = \frac{a}{2a-8}$

$$\nabla^2 f(x) = \begin{bmatrix} 2 & -4 \\ -4 & 2a \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & -4 \\ 0 & 2a-8 \end{bmatrix}$$

$2a-8 > 0$ if $a > 4$ so local minimizer at $\left(\frac{a}{2a-8}, \frac{1}{a-4}\right)$ for all $a > 4$.

3. Find the local minimizers of

$$f(x) = f(x_1, x_2, x_3) = x_1^2 - x_2 x_3$$

subject to

$$x_1 + x_2 + 2x_3 = 1$$

$x_1 + x_2 + 2x_3 = 0 \quad x_1 = -x_2 - 2x_3 \quad \begin{bmatrix} -x_2 - 2x_3 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} x_2 + \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} x_3$

$$Z = \begin{bmatrix} -1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$Z^T \nabla f(x) = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2x_1 \\ -x_3 \\ -x_2 \end{bmatrix} = \begin{bmatrix} -2x_1 - x_3 \\ -4x_1 - x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x_2 = -4x_1 \quad x_3 = -2x_1 \quad x_1 - 4x_1 - 2(2x_1) = -7x_1 = 1$

$x_1 = -1/7, \; x_2 = 4/7, \; x_3 = 2/7$

$$Z^T \nabla^2 f(x) Z = \begin{bmatrix} -1 & 1 & 0 \\ -2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} -1 & -2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 2 & 3 \\ 3 & 8 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 2 & 3 \\ 0 & 8 - 9/2 \end{bmatrix}$$

pos def

So $(-1/7, 4/7, 2/7)$ is the only local minimizer

4. (a) Prove that if A is $m \times n$, $A\bar{x} = b$ and $Ax = b$, then $x = \bar{x} + Zv$ for some $v \in \mathbf{R}^{n-m}$, where Z is a basis null space matrix for A. (b) Given

$$A = \begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix} \qquad b = \begin{bmatrix} 3 \\ -4 \end{bmatrix} \qquad \bar{x} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} \qquad x = \begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix}$$

find v.

[10 points] (a) $A(x-\bar{x}) = Ax - A\bar{x} = b - b = 0$ so $x - \bar{x} \in N(A) = R(Z)$ and thus $x - \bar{x} = Zv$ for some $v \in \mathbb{R}^{n-m}$, That is $x = \bar{x} + Zv$.

[10 points] (b) $\begin{bmatrix} 0 & 1 & -1 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad p_2 - p_3 = 0$, $-p_1 + 2p_3 = 0$

$p_2 = p_3 \quad p_1 = 2p_3 \qquad Z = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$

$$x = \begin{bmatrix} -2 \\ 0 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} v \qquad v = -2$$

5. Consider the problem

Minimize $f(x) = f(x_1, \overset{x_3}{\cancel{x_2}}) = x_1 - x_2$
subject to

$$\begin{aligned} x_1 \qquad + x_3 &= 1 \\ -x_1 + x_2^2 \qquad &= 2 \end{aligned}$$

Show that $(-7/4, 1/2, 11/4)^T$ is a local minimizer by demonstrating that $(-7/4, 1/2, 11/4)^T$ satisfies the sufficient conditions presented in the course.

$-\frac{7}{4} + \frac{11}{4} = 1 \quad \frac{7}{4} + \left(\frac{1}{2}\right)^2 = 2$ (feasible)

$\nabla g(x)^T = \begin{bmatrix} 1 & -1 \\ 0 & 2x_2 \\ 1 & 0 \end{bmatrix} \quad \nabla g(-7/4, 1/2, 11/4)^T = \begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 1 & 0 \end{bmatrix}$ rank = 2 (regular)

$\mathcal{L}(x, \lambda) = x_1 - x_2 - \lambda_1(x_1 + x_3 - 1) + \lambda_2(-x_1 + x_2^2 - 2)$

$\nabla_x \mathcal{L}(x, \lambda) = \begin{bmatrix} 1 - \lambda_1 - \lambda_2 \\ -1 - 2\lambda_2 x_2 \\ -\lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \quad \lambda_1 = 0 \quad \lambda_2 = -1$

$-1 - 2(-1)\left(\frac{1}{2}\right) = 0$

$\nabla(-7/4, 1/2, 11/4)^T p = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad \begin{aligned} p_1 + p_3 &= 0 \\ -p_1 + p_2 &= 0 \end{aligned}$

$Z = Z(-7/4, 1/2, 11/4) = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}$

$\nabla^2_{xx} \mathcal{L}(x, \lambda) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & -2\lambda_2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad \nabla^2_{xx} \mathcal{L}(-7/4, 1/2, 11/4, 0, -1) = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$

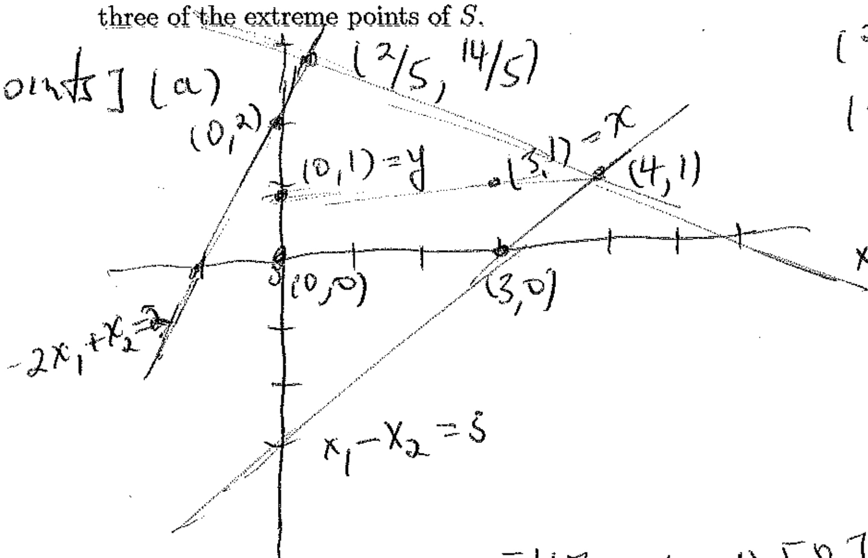
$Z^T \nabla^2_{xx} \mathcal{L}(-7/4, 1/2, 11/4, 0, -1) Z = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix} = 2 > 0$

6. Consider the convex set S defined by the inequalities

$$\begin{aligned} -2x_1 + x_2 &\le 2 \\ x_1 - x_2 &\le 3 \\ x_1 + 2x_2 &\le 6 \\ x_1, x_2 &\ge 0 \end{aligned}$$

(a) Find the extreme points of S. (b) Write $x = (3,1)^T$ as a convex combination of three of the extreme points of S.

[10 points] (a)



$(2/5, 14/5), (4,1),$
$(3,0), (0,0), (0,2)$

[10 points] (b) $x = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = t\begin{bmatrix} 4 \\ 1 \end{bmatrix} + (1-t)\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{3}{4}\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \frac{1}{4}y$

$$y = \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

$$\begin{aligned} x = \begin{bmatrix} 3 \\ 1 \end{bmatrix} &= \frac{3}{4}\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \frac{1}{4}\left(\frac{1}{2}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 0 \\ 2 \end{bmatrix}\right) \\ &= \frac{3}{4}\begin{bmatrix} 4 \\ 1 \end{bmatrix} + \frac{1}{8}\begin{bmatrix} 0 \\ 0 \end{bmatrix} + \frac{1}{8}\begin{bmatrix} 0 \\ 2 \end{bmatrix} \end{aligned}$$

(There are other solutions.)

7. Find the local minimizers of

$$f(x) = f(x_1, x_2) = x_1^3 - x_2^3 - x_1^2 + 2x_1 + 2x_2 + 1$$
subject to

$$\begin{aligned} 2x_1 - x_2 &\geq 1 \\ x_1 &\geq 0 \end{aligned}$$

(a) if both constraints are active (b) if both constraints are inactive.

[15 points] (a) $2x_1 - x_2 = 1$ $x_1 = 0$ so $x_2 = -1$

$$\nabla f(x) = \begin{bmatrix} 3x_1^2 - 2x_1 + 2 \\ -3x_2^2 + 2 \end{bmatrix} \qquad A = \begin{bmatrix} 2 & -1 \\ 1 & 0 \end{bmatrix} \quad A^T = \begin{bmatrix} 2 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\nabla f(0,-1) = \begin{bmatrix} 2 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}\lambda_1 + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\lambda_2 \qquad \lambda_1 = 1, \ \lambda_2 = 0$$

$$\hat{A}_+ = [2 \ \ -1] \qquad Z_+ = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \nabla^2 f(x) = \begin{bmatrix} 6x_1 - 2 & 0 \\ 0 & -6x_2 \end{bmatrix}$$

$$Z_+^T \nabla^2 f(0,-1) Z_+ = [1 \ \ 2] \begin{bmatrix} -2 & 0 \\ 0 & 6 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 22 > 0$$

so $(0,-1)$ is a local minimizer.

[5 points] (b) $\lambda_1 = \lambda_2 = 0$

$$\begin{bmatrix} 3x_1^2 - 2x_1 + 2 \\ -3x_2^2 + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad 3x_1^2 - 2x_1 + 2 = 0$$

$$x_1 = \frac{2 \pm \sqrt{4 - 24}}{6} \qquad \text{no solution}$$

so no local minimizers

8. Let A be an $m \times n$ matrix. Prove that if the linear programming problem: Minimize $z = c^T x$ subject to $Ax \geq b$ is feasible but unbounded, then there is no solution $y \in \mathbf{R}^m$ to the linear system $A^T y = c$ such that $y \geq 0$. (Hint: Consider the dual.)

The dual is Maximize $w = b^T y$
subject to $A^T y = c$
$y \geq 0$

By a corollary to weak duality, since the problem is feasible but unbounded, its dual is infeasible, that is, there is no solution to $A^T y = c$ such that $y \geq 0$.

9. Consider the problem

Minimize $f(x) = f(x_1, x_2) = 2x_1^2 + x_2 + 3$
Subject to

$$\begin{aligned} x_2 &\geq x_1^2 - 1 \\ x_1 &\geq x_2 \end{aligned}$$

Show that $(0, -1)^T$ is a local minimizer by demonstrating that $(0, -1)^T$ satisfies the sufficient conditions presented in the course.

$g_1(x) = -x_1^2 + x_2 + 1 \geq 0 \quad g_2(x) = x_1 - x_2 \geq 0$

$g_1(0,-1) = 0$ active $\quad g_2(0,-1) = 1 > 0$ inactive so $\lambda_2 = 0$

$\mathcal{L}(x, \lambda) = 2x_1^2 + x_2 + 3 - \lambda_1(-x_1^2 + x_2 + 1)$

$\nabla_x \mathcal{L}(x, \lambda) = \begin{bmatrix} 4x_1 + 2\lambda_1 x_1 \\ 1 - \lambda_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ so $\lambda_1 = 1 > 0$

$\nabla g_1(x) = \begin{bmatrix} -2x_1 \\ 1 \end{bmatrix} \quad \nabla g_1(0,-1) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ rank $= 1$ so $(0,-1)^T$ regular

$[0 \quad 1] \begin{bmatrix} p_1 \\ p_2 \end{bmatrix} = 0 \quad p_2 = 0$ so $Z(0,-1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\nabla^2_{xx} \mathcal{L}(x, \lambda) = \begin{bmatrix} 4 + 2\lambda_1 & 0 \\ 0 & 0 \end{bmatrix} \quad \nabla^2_{xx} \mathcal{L}(x, \lambda) = \begin{bmatrix} 6 & 0 \\ 0 & 0 \end{bmatrix}$

$Z(0,-1)^T \nabla^2_{xx} \mathcal{L}(0, -1, 1, 0) Z(0,-1) = 6 > 0$

10. Let $x = x_* + \gamma p \in \mathbf{R}^n$. Prove that if $p \in \mathbf{R}^n$ such that $p^T \nabla f(x_*) < 0$, then $f(x) < f(x_*)$ for all $\gamma > 0$ sufficiently small. (Hint: Write the remainder form of the quadratic Taylor polynomial.)

$$f(x) = f(x_* + \gamma p) = f(x_*) + \gamma p^T \nabla f(x_*) + \frac{1}{2} \gamma p^T \nabla^2 f(\xi) \gamma p$$

$$= f(x_*) + \gamma \left(p^T \nabla f(x_*) + \frac{\gamma}{2} p^T \nabla^2 f(\xi) p \right)$$

If $p^T \nabla^2 f(\xi) p < 0$ then $f(x) < f(x_*)$ for all $\gamma > 0$ since $p^T \nabla f(x_*) < 0$.

If $p^T \nabla^2 f(\xi) p > 0$ then

$$p^T \nabla f(x_*) + \frac{\gamma}{2} p^T \nabla^2 f(\xi) p < 0$$

and thus $f(x) < f(x_*)$ for all $\gamma > 0$ such that

$$\gamma < \frac{-p^T \nabla f(x_*)}{\frac{1}{2} p^T \nabla^2 f(\xi) p}$$