1. Express $(2,2)^T$ as a convex combination of $(0,1)^T, (1,4)^T$ and $(3,1)^T$.

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = a\begin{bmatrix} 0 \\ 1 \end{bmatrix} + b\begin{bmatrix} 1 \\ 4 \end{bmatrix} + c\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

$$2 = b + 3c \qquad b = 2 - 3c$$

$$2 = a + 4b + c \qquad 2 = a + 4(2-3c) + c$$

$$1 = a + b + c \qquad 1 = a + (2 - 3c) + c$$

$$2 = a - 11c + 8$$

$$1 = a - 2c + 2$$

$$1 = -9c + 6 \qquad c = 5/9 \qquad b = 2 - \frac{15}{9} = \frac{3}{9}$$

so $a = 1/9$

$$\begin{bmatrix} 2 \\ 2 \end{bmatrix} = \frac{1}{9}\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \frac{3}{9}\begin{bmatrix} 1 \\ 4 \end{bmatrix} + \frac{5}{9}\begin{bmatrix} 3 \\ 1 \end{bmatrix}$$

2. In solving a linear programming problem by the simplex method, we arrive at the objective function

$$z = -2x_3 + 3x_4 + 4x_5 + 7$$

and the dictionary

$$\begin{aligned} x_1 &= -3x_3 - 3x_4 + x_5 + 4 \\ x_2 &= -x_3 + x_4 + 2x_5 + 1 \end{aligned}$$

Continue the simplex method to find the optimal solution to this problem.

x_3 in so $x_4 = x_5 = 0$

$x_1 = -3x_3 + 4 \geq 0$ $\quad x_3 \leq 4/3$

$x_2 = -x_3 + 1 \geq 0$ $\quad x_3 \leq 1$

so $x_3 = 1$ and x_2 out

$x_3 = -x_2 + x_4 + 2x_5 + 1$

$z = -2(-x_2 + x_4 + 2x_5 + 1) + 3x_4 + 4x_5 + 7$

$= 2x_2 + x_4 + 5$

so the basis $\{x_1, x_3\}$ is optimal with solution $x_1 = -3(1) + 4 = 1$ thus $(1, 0, 1, 0, 0)$ and Min $z = 5$

3. Consider the following set of constraints.

$$\begin{aligned} x_1 + x_2 - x_3 &\geq 2 \\ 2x_1 \quad\quad - x_3 &\geq 3 \\ x_2 + 4x_3 &\geq 1 \\ x_1 + 3x_2 + x_3 &\geq 4 \end{aligned}$$

(a) Show that $\bar{x} = (2, 2, 1)^T$ is feasible and indicate which constraints are active and which are inactive. (b) Show that $p = (1, -3, 1)^T$ is a feasible direction at $\bar{x}$. (c) Find the largest value $\bar{\alpha}$ among all $\alpha \geq 0$ such that $\bar{x} + \alpha p$ is feasible.

[7 points](a) $2 + 2 - 1 = 3 > 2$ inactive

$4 \quad -1 = 3 = 3$ active

$2 + 4 = 6 > 1$ inactive

$2 + 6 + 1 = 9 > 4$ inactive

[3 points] (b) $\begin{bmatrix} 2 & 0 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = 1 \geq 0$ so feasible

[10 points] (c) $\bar{x} + \alpha p = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ -3 \\ 1 \end{bmatrix} = \begin{bmatrix} 2+\alpha \\ 2-3\alpha \\ 1+\alpha \end{bmatrix}$

$\begin{bmatrix} 1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2+\alpha \\ 2-3\alpha \\ 1+\alpha \end{bmatrix} = 2 + \alpha + 2 - 3\alpha - (1+\alpha)$
$= 3 - 3\alpha \geq 2$ if $\alpha \leq 1/3$

$\begin{bmatrix} 0 & 1 & 4 \end{bmatrix} \begin{bmatrix} 2+\alpha \\ 2-3\alpha \\ 1+\alpha \end{bmatrix} = 2 - 3\alpha + 4 + 4\alpha = 6 + \alpha \geq 1$
for all α

$\begin{bmatrix} 1 & 3 & 1 \end{bmatrix} \begin{bmatrix} 2+\alpha \\ 2-3\alpha \\ 1+\alpha \end{bmatrix} = 2 + \alpha + 6 - 9\alpha + 1 + \alpha$
$= 9 - 7\alpha \geq 4$ if $\alpha \leq 5/4$

so $\bar{\alpha} = 1/3$

4. Given the linear programming problem below, do the following: (a) Write the problem in canonical form. (b) Write the dual problem with objective function $w = g(y)$. (c) Prove that the dual problem has no optimal solution by finding feasible vectors $y^{(n)}$ such that $\lim_{n\to\infty} g(y^{(n)}) = \infty$.

Minimize $z = f(x) = 4x_1 - x_2$

subject to

$$\begin{aligned} x_1 + \ x_2 &\le 6 \\ x_1 - \ x_2 &\ge 3 \\ -x_1 + 2x_2 &\ge 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

[3 points] (a) Minimize $z = f(x) = 4x_1 - x_2$

subject to $-x_1 - x_2 \ge -6$

$$\begin{aligned} x_1 - x_2 &\ge 3 \\ -x_1 + 2x_2 &\ge 2 \\ x_1, x_2 &\ge 0 \end{aligned}$$

[7 points] (b) Maximize $w = g(y) = -6y_1 + 3y_2 + 2y_3$

subject to $-y_1 + y_2 - y_3 \le 4$

$$\begin{aligned} -y_1 - y_2 + 2y_3 &\le -1 \\ y_1, y_2, y_3 &\ge 0 \end{aligned}$$

[10 points] (c) Let $y^{(n)} = (n, 2n, n)^T$

$-n + 2n - n = 0 \le 4$

$-n - 2n + 2n = -n \le -1$

so $y^{(n)}$ is feasible

$w = g(y^{(n)}) = 4n - 2n = 2n$

so $\lim_{n\to\infty} g(y^{(n)}) = \infty$

5. Suppose Minimize $z = c^T x$ subject to $Ax = b, x \geq 0$ is a linear programming problem such that the feasible set S is bounded. Then, by the Representation Theorem, if $x \in S$ then x is a convex combination of the extreme points $\{v_1, \ldots, v_k\}$ of S. The problem has an optimal soution. Prove that one of the extreme points $v_1, \ldots, v_k$ is an optimal solution to the problem.

Let $x_* = \sum_{i=1}^{k} \alpha_i v_i$ be an optimal solution

If no v_i is optimal, then $c^T x_* < c^T v_i$ for all i. But

$$c^T x_* = c^T \left(\sum_{i=1}^{k} \alpha_i v_i \right) = \sum_{i=1}^{k} \alpha_i c^T v_i$$

$$< \sum_{i=1}^{k} \alpha_i c^T x_* = c^T x_* \sum_{i=1}^{k} \alpha_i = c^T x_*$$

Contradiction, so $c^T v_i = c^T x_*$ for some i.