1. Find all the local minimizers of the function

$$f(x) = f(x_1, x_2) = x_1^3 + x_2^3 - 3x_1x_2 + 3.$$

$$\nabla f(x_1,x_2) = \begin{bmatrix} 3x_1^2 - 3x_2 \\ 3x_2^2 - 3x_1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \qquad \begin{matrix} x_1^2 = x_2 \\ x_2^2 = x_1 \end{matrix}$$

$x_1 = x_2^2 = (x_1^2)^2 = x_1^4 \quad x_1(x_1^3 - 1) = 0$ so $x_1 = 0, 1$

stationary points $(0,0), (1,1)$

$$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 6x_1 & -3 \\ -3 & 6x_2 \end{bmatrix}$$

$$\nabla^2 f(0,0) = \begin{bmatrix} 0 & -3 \\ -3 & 0 \end{bmatrix} \quad \det \begin{bmatrix} -\lambda & 3 \\ 3 & \lambda \end{bmatrix} = \lambda^2 - 9 = 0$$

$\lambda = \pm 3$ so not pos def.

$$\nabla^2 f(1,1) = \begin{bmatrix} 6 & -3 \\ -3 & 6 \end{bmatrix} \rightsquigarrow \begin{bmatrix} 6 & -3 \\ 0 & 6 - 3/2 \end{bmatrix}$$ pos def

so $(1,1)$ local minimizer

2. Use Newton's method to find a stationary point of

$$f(x) = f(x_1, x_2) = \tfrac{3}{2}x_1^2 + x_1x_2 + x_2^2 - x_1 + 2x_2 + 5$$

starting with $x_0 = (-3, 1)^T$.

$$\nabla f(x) = \begin{bmatrix} 3x_1 + x_2 - 1 \\ x_1 + 2x_2 + 2 \end{bmatrix} \qquad \nabla^2 f(x) = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}$$

$$\nabla f(-3, 1) = \begin{bmatrix} -9 \\ 1 \end{bmatrix} \qquad \nabla^2 f(x)^{-T} = \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}$$

$$x_1 = \begin{bmatrix} -3 \\ 1 \end{bmatrix} - \frac{1}{5}\begin{bmatrix} 2 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -9 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \end{bmatrix} - \begin{bmatrix} -19/5 \\ 12/5 \end{bmatrix} = \begin{bmatrix} 4/5 \\ -7/5 \end{bmatrix}$$

$$\nabla f(4/5, -7/5) = \begin{bmatrix} \frac{12}{5} - \frac{7}{5} + 1 \\ \frac{4}{5} - \frac{14}{5} + 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$x = (4/5, -7/5)^T$ is a stationary point

3. Consider the problem

Minimize $f(x_1, x_2) = 4x_1 + x_2$
subject to

$$\begin{aligned} (x_1+1)^2 + x_2^2 &\geq 1 \\ x_1^2 + x_2^2 &\leq 2 \end{aligned}$$

(a) Write the Lagrangian function $\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2)$ of this problem. (b) Show that $(1,1)^T$ is a regular point. (c) Find the Lagrange multipliers λ_1 and λ_2 such that $(1, 1, \lambda_1, \lambda_2)$ is a stationary point of $\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2)$.

[3 points] (a) $\mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = 4x_1 + x_2 - \lambda_1((x_1+1)^2 + x_2^2 - 1)$
$- \lambda_2(-x_1^2 - x_2^2 + 2)$

[7 points] (b) $g(x) = [-x_1^2 - x_2^2 + 2]$ (active)

$\nabla g(x)^T = [-2x_1 \;\; -2x_2]$ $\quad \nabla g(1,1)^T = [-2 \;\; -2]$

[10 points] (c)

$$\nabla_x \mathcal{L}(x_1, x_2, \lambda_1, \lambda_2) = \begin{bmatrix} 4 - 2\lambda_1(x_1+1) + 2\lambda_2 x_1 \\ 1 - 2\lambda_1 x_2 + 2\lambda_2 x_2 \end{bmatrix}$$

$$\nabla_x \mathcal{L}(1, 1, \lambda_1, \lambda_2) = \begin{bmatrix} 4 - 4\lambda_1 + 2\lambda_2 \\ 1 - 2\lambda_1 + 2\lambda_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$3 - 2\lambda_1 = 0 \quad \underline{\lambda_1 = 3/2} \quad -2 + 2\lambda_2 = 0 \quad \underline{\lambda_2 = 1}$

4. Consider the linear programming problem

Minimize $z = -2x_1 + 3x_2 - x_3 + x_4$
subject to

$$\begin{aligned} x_1 + 2x_2 \qquad - 3x_4 &= 3 \\ -x_1 + x_2 + x_3 + 3x_4 &= -2 \\ x_1, x_2, x_3, x_4 &\geq 0 \end{aligned}$$

(a) Find variables x_i and x_j such that $\{x_i, x_j\}$ is not a basis. (b) Find a basis $\{x_i, x_j\}$ such that the basic solution is not feasible. (c) Find all bases $\{x_i, x_j\}$ such that the basic solution is feasible. (d) Use part (c) to solve the problem.

[5 points] (a)

$$A = \begin{array}{c} \begin{array}{cccc} x_1 & x_2 & x_3 & x_4 \end{array} \\ \begin{bmatrix} 1 & 2 & 0 & -3 \\ -1 & 1 & 1 & 3 \end{bmatrix} \end{array} \qquad (x_1, x_4)$$

[5 points] (b) $\{x_3, x_4\}$ $-3x_4 = 3$ so $x_4 = -1$, $x_3 = 1$

(or $\{x_2, x_3\}$, $\{x_2, x_4\}$)

[5 points] (c) $\{x_1, x_2\}$ $x = (7/3, 1/3, 0, 0)$

$\{x_1, x_3\}$ $x = (3, 0, 1, 0)$

[5 points] (d) $z(7/3, 1/3, 0, 0) = -2(7/3) + 3(1/3) = -11/3$

$z(3, 0, 1, 0) = -2(3) + 1 = -5$

so $x = (3, 0, 1, 0)$ is the optimal solution

5. Show that $(2,1)^T$ is a local minimizer for the problem

$$\begin{aligned} \text{Minimize } & f(x_1,x_2) = 2x_1^2 - 2x_1x_2 + x_2^2 - 2x_1 + x_2 \\ \text{subject to} & \\ & x_1^2 - x_2 = 3 \end{aligned}$$

by demonstrating that $(2,1)^T$ satisfies the sufficient conditions presented in the course.

$(2)^2 - 1 = 3$ so $(2,1)^T$ is feasible

$$\mathcal{L}(x_1,x_2,\lambda) = 2x_1^2 - 2x_1x_2 + x_2^2 - 2x_1 + x_2 - \lambda(x_1^2 - x_2)$$

$$\nabla_x \mathcal{L}(x_1,x_2,\lambda) = \begin{bmatrix} 4x_1 - 2x_2 - 2 - 2\lambda x_1 \\ -2x_1 + 2x_2 + 1 + \lambda \end{bmatrix}$$

$$\nabla_x \mathcal{L}(2,1,\lambda) = \begin{bmatrix} 8-2-2-4\lambda \\ -4+2+1+\lambda \end{bmatrix} = \begin{bmatrix} 4-4\lambda \\ -1+\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

if $\lambda = 1$. $\nabla g(x_1,x_2) = \begin{bmatrix} 2x_1 \\ 1 \end{bmatrix}$ $\nabla g(2,1)^T = [4 \ \ -1]$

so $(2,1)$ is regular

$[4 \ \ -1]\begin{bmatrix} u \\ v \end{bmatrix} = 4u - v = 0$ $v = 4u$ $Z(2,1) = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$

$$\nabla^2_{xx} \mathcal{L}(x_1,x_2,\lambda) = \begin{bmatrix} 4-2\lambda & -2 \\ -2 & 2 \end{bmatrix} \quad \nabla^2_{xx}\mathcal{L}(2,1,1) = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}$$

$$Z(2,1)^T \nabla^2_{xx} \mathcal{L}(2,1,1) Z(2,1)$$

$$= [1 \ \ 4]\begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 4 \end{bmatrix} = [-6 \ \ 6]\begin{bmatrix} 1 \\ 4 \end{bmatrix} = 18 > 0$$

so positive definite

6. Let x_* be a feasible solution to the problem: Minimize $c^T x$ subject to $Ax = b, x \geq 0$ and let y_* be a feasible solution to is dual: Maximize $b^T y$ subject to $A^T y \leq c$. Prove that if $x_*^T(c - A^T y_*) = 0$, then x_* and y_* are optimal solutions to their respective problems.

By hypothesis $Ax_* = b$ and $x_*^T c = x_*^T A^T y_*$

$$c^T x_* = (c^T x_*)^T = x_*^T c_* = x_* A^T y_*$$
$$= (A x_*)^T y_* = b^T y_*$$

Since $c^T x_*$ is 1-by-1 and x_*, y_* are optimal by a Corollary.

7. Given the matrix

$$A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix}$$

(a) Find a basis null space matrix Z for A. (b) Let $x = (3, -1, -2)^T$. Find vectors v and λ such that $x = Zv + A^T\lambda$.

[7 points] (a) $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ $\quad u - w = 0 \quad u = w$

$v + 2w = 0 \quad v = -2w$

$$\begin{bmatrix} w \\ -2w \\ w \end{bmatrix} = w \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \text{ so } Z = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

[13 points] (b) $\begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} v + \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} \lambda_1 \\ \lambda_2 \end{bmatrix} = \begin{bmatrix} v + \lambda_1 \\ -2v + \lambda_2 \\ v - \lambda_1 + 2\lambda_2 \end{bmatrix}$

$\lambda_1 = 3 - v \quad \lambda_2 = -1 + 2v$

$v - (3 - v) + 2(-1 + 2v) = -2 \qquad 6v - 5 = -2$

so $\underline{v = 1/2}$, $\underline{\lambda_1 = 5/2}$, $\underline{\lambda_2 = 0}$

8. Consider the problem

Minimize $f(x_1, x_2) = \frac{1}{2}x_1^2 + x_2^2$
subject to

$$\begin{array}{rcrl} 2x_1 + x_2 & \geq & 2 & [1] \\ -x_1 + x_2 & \geq & -1 & [2] \\ x_1 & \geq & 0 & [3] \end{array}$$

Find all the local minimizers with the property that exactly one of the constraints [1], [2] or [3] is active.

$$\nabla f(x_1, x_2) = \begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix} = A^T\lambda = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\lambda_1 + \begin{bmatrix} -1 \\ 1 \end{bmatrix}\lambda_2 + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\lambda_3$$

[1] active $\lambda_2 = \lambda_3 = 0$, $2x_1 + x_2 = 2$ $\quad x_2 = 2 - 2x_1$

$$\begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\lambda_1 \quad \begin{array}{l} x_1 = 2\lambda_1 \\ 2(2-2x_1) = \lambda_1 \end{array} \quad \begin{array}{l} 4 - 4x_1 = \lambda_1 \\ 4 - 8\lambda_1 = \lambda_1 \end{array}$$

$\lambda_1 = 4/9 > 0$ $\quad x_1 = 8/9$, $x_2 = 2 - 2(8/9) = 2/9$

$\nabla^2 f(x_1, x_2) = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ pos. def. so $(8/9, 2/9)$ local min.

[2] active $\lambda_1 = \lambda_3 = 0$ $\quad -x_1 + x_2 = -1$ $\quad x_2 = x_1 - 1$

$$\begin{bmatrix} x_1 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} x_1 \\ 2x_1 - 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}\lambda_2 \quad \begin{array}{l} x_1 = -\lambda_2 \\ 2x_1 - 2 = \lambda_2 \end{array}$$

$3x_1 - 2 = 0$ $\quad x_1 = 2/3$ then $\lambda_2 = -2/3 < 0$ so no local min.

[3] active $\lambda_1 = \lambda_2 = 0$ $\quad x_1 = 0$

$$\begin{bmatrix} 0 \\ 2x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}\lambda_3 \quad x_2 = 0 \text{ but } 2x_1 + x_2 \geq 2$$

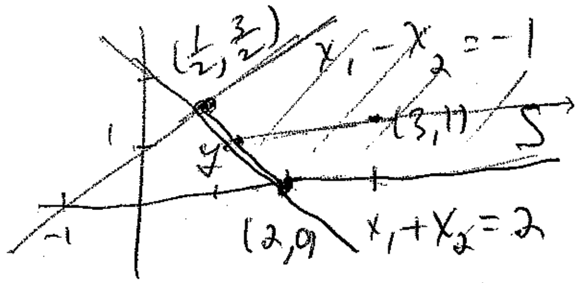
fails so $(0, 0)$ is not feasible.

9. Let S be the set of vectors $x = (x_1, x_2)^T$ that satisfy the conditions

$$\begin{aligned} x_1 + x_2 &\geq 2 \\ -x_1 + x_2 &\leq 1 \\ x_1, x_1 &\geq 0 \end{aligned}$$

$x_1 - x_2 \geq -1$

(a) Show that $d = (1, 0)^T$ is a direction of unboundedness for S. (b) Write $x = (3, 1)^T$ as the sum of a scalar multiple of d and a convex combination of extreme points of S.



[5 points] (a) $A = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$

$$Ad = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \geq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

[15 points] (b) $\begin{bmatrix} 3 \\ 1 \end{bmatrix} + \gamma \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3+\gamma \\ 1 \end{bmatrix}$ $\quad (3+\gamma) + 1 = 2$ if $\gamma = -2$

so $y = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} = t \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} + (1-t) \begin{bmatrix} 2 \\ 0 \end{bmatrix} \quad \tfrac{3}{2} t = 1 \quad t = 2/3$$

$$x = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + \alpha \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$3 = \frac{1}{3} + \frac{2}{3} + \alpha(1)$ so $\alpha = 2$

$$x = \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \frac{2}{3} \begin{bmatrix} 1/2 \\ 3/2 \end{bmatrix} + \frac{1}{3} \begin{bmatrix} 2 \\ 0 \end{bmatrix} + 2 \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

10. Consider the problem: Minimize $f(x)$ (nonlinear) subject to $Ax = b$. (a) Define the reduced function $\phi(v)$ for this problem. (b) Write the equation that relates $\nabla\phi(v)$ to $\nabla f(x)$. (c) Now prove that if, for a vector x_*, there is a vector λ_* such that $\nabla f(x_*) = A^T\lambda_*$, then $\nabla\phi(v_*) = 0$, where v_* is related to x_* by the definition of $\phi(v)$.

[5 points] (a) $\varphi(v) = f(\bar{x} + Zv)$ where

$A\bar{x} = b$ and $R(Z) = N(A)$

[5 points] (b) $\nabla\varphi(v) = Z^T \nabla f(x)$ where $x = \bar{x} + Zv$

[10 points] (c) $\nabla\varphi(v_*) = Z^T \nabla f(x_*) = Z^T A^T \lambda_* = 0$

because $Z^T A^T = (AZ)^T$ and $R(Z) = N(A)$

implies that $AZ = 0$