1. Convert the linear programming problem below to *canonical* form.

minimize $3x_1 - 2x_3$
subject to

$$
\begin{aligned}
x_1 - 2x_2 + x_3 &= 1 \\
x_1 + x_2 &\geq 4 \\
x_1, x_2 &\geq 0 \quad , x_3 \leq 3
\end{aligned}
$$

$x_3' = 3 - x_3$, $x_3 = 3 - x_3'$

min $3x_1 - 2(3 - x_3')$

sub to $x_1 - 2x_2 + (3 - x_3') = 1$

$x_1 + x_2 \geq 4$

$x_1, x_2, x_3' \geq 0$

min $3x_1 + 2x_3' - 6$

sub to $x_1 - 2x_2 - x_3' = -2$

$x_1 + x_2 \geq 4$

$x_1, x_2, x_3' \geq 0$

min $3x_1 + 2x_3'$

sub to $x_1 - 2x_2 - x_3' \geq -2$

$-x_1 + 2x_2 + x_3' \geq 2$

$x_1 + x_2 \geq 4$

$x_1, x_2, x_3' \geq 0$

2. Given the linear programming problem

minimize $z = x_1 - x_2$
subject to

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &\geq 2 \\ x_1 + 2x_2 - x_3 &\geq 1 \\ x_1, x_2, x_3 &\geq 0 \end{aligned}$$

(5 points each)

(a) Show that $x = (2, 0, 1)^T$ is a feasible solution to the problem.
(b) Show that $p = (-1, 2, 1)^T$ is a feasible direction at the feasible solution $x = (2, 0, 1)^T$.
(c) Determine the maximal step length $\alpha \geq 0$ such that $x + \alpha p$ remains feasible, where x and p are as in part (b).
(d) Find *all* the feasible directions $p = (p_1, p_2, p_3)^T$ at $x = (2, 0, 1)^T$.

(a) $2 - 0 + 3 > 2$ inactive $\quad X \geq 0$

$2 + 0 - 1 = 1$ active

(b) $\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = -1 + 4 - 1 \geq 0 \qquad p_2 = 2 \geq 0$

(c) $\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} 2 + \alpha(1) \\ 0 + \alpha(2) \\ 1 + \alpha(1) \end{bmatrix} \geq 2$

$2 + \alpha - 4\alpha + 3 + 3\alpha \geq 2$

$5 - 2\alpha \geq 2 \qquad 3 \geq 2\alpha \qquad 0 \leq \alpha \leq 3/2$

(d) $\begin{bmatrix} 1 & 2 & -1 \end{bmatrix} \begin{bmatrix} p_1 \\ p_2 \\ p_3 \end{bmatrix} = p_1 + 2p_2 - p_3 \geq 0$

$p_2 \geq 0$

3. Given that the linear programming problem

(1) maximize $z = c^T x$

subject to $Ax \le b$

is feasible but unbounded, prove that if the linear programming problem

(2) minimize $z = c^T x$

subject to $Ax \ge b$

is feasible, then it is also unbounded. (Hint: Write the duals.)

Dual of (1): min $b^T y$
sub to $A^T y = c$, $y \ge 0$

(2) same as max $-c^T x$
sub to $-Ax \le -b$ so dual
of (2) min $-b^T y$ sub to $-A^T y = -c$, $y \ge 0$
same as $A^T y = c$, $y \ge 0$

Since (1) is unbounded, dual is
infeasible: no $y \ge 0$ with $A^T y = c$ so dual
of (2) is infeasible, then (2) is unbounded.

4. Given the linear programming problem

minimize $z = x_1 - 3x_2 + x_3 + 4x_4 - 2x_5$
subject to

$$\begin{aligned} x_1 + 2x_2 \qquad - x_4 + x_5 &= 1 \\ 2x_2 + x_3 - x_4 + 2x_5 &= 2 \\ x_1, x_2, x_3 . x_4, x_5 &\geq 0 \end{aligned}$$

(3 points) (a) Find a pair of variables that are not a basis.
(5 points) (b) Find a basis for which the basic solution is not feasible.
(12 points) (c) Show that the basis $\{x_3, x_5\}$ is optimal, that is, its basic solution is an optimal solution, without finding any other basic solutions.

(a) $\{x_2, x_4\}$ since $\det \begin{bmatrix} 2 & -1 \\ 2 & -1 \end{bmatrix} = 0$

(b) $\{x_1, x_4\}$ basic solution $(-1, 0, 0, -2, 0)$
(not the only correct answer)

(c) $x_5 = 1 - x_1 - 2x_2 + x_4$

$x_3 = 2 - 2x_2 + x_4 - 2(1 - x_1 - 2x_2 + x_4)$

$= 2x_1 + 2x_2 - x_4$

$z = x_1 - 3x_2 + (2x_1 + 2x_2 - x_4)$

$+ 4x_4 - 2(1 - x_1 - 2x_2 + x_4)$

$= 5x_1 + 3x_2 + x_4 - 2$

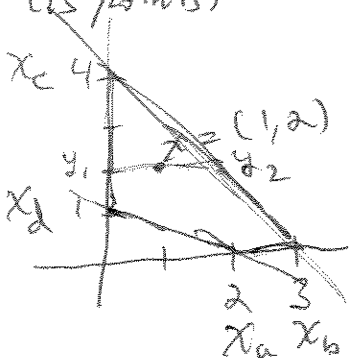
coefficients of nonbasic
variables > 0.

5. Let S be the convex set of points $x = (x_1, x_2)$ that satisfy

$$\begin{aligned} x_1 + 2x_2 &\geq 2 \\ 4x_1 + 3x_2 &\leq 12 \\ x_1, x_2 &\geq 0 \end{aligned}$$

(5 points) (a) List all the extreme points of S, calling them x_a, x_b and so on.

(15 points) (b) Write $x = (1, 2)$ as a convex combination of extreme points of S.



(a) $x_a\ (2,0)$ $x_b\ (3,0)$ $x_c\ (0,4)$ $x_d\ (0,1)$

(b) There are many solutions; here's mine.

$y_1 = (0,2)$ $4x_1 + 3(2) = 12$ $x_1 = 3/2$

so $y_2 = (3/2, 2)$

$(0,2) = a(0,1) + (1-a)(0,4)$

$2 = a + 4 - 4a$ $3a = 2$

$y_1 = \frac{2}{3} x_d + \frac{1}{3} x_c$

$(3/2, 2) = a(3,0) + (1-a)(0,4)$

$3/2 = 3a$ $a = 1/2$

$y_2 = \frac{1}{2} x_b + \frac{1}{2} x_c$

$$\begin{aligned} x &= \tfrac{1}{2} y_1 + \tfrac{1}{2} y_2 \\ &= \tfrac{1}{2}\left(\tfrac{2}{3} x_d + \tfrac{1}{3} x_c\right) + \tfrac{1}{2}\left(\tfrac{1}{2} x_b + \tfrac{1}{2} x_c\right) \\ &= \tfrac{1}{3} x_d + \left(\tfrac{1}{6} + \tfrac{1}{4}\right) x_c + \tfrac{1}{4} x_b \\ &= \tfrac{1}{3} x_d + \tfrac{5}{12} x_c + \tfrac{1}{4} x_b \end{aligned}$$

(Clever answer: $x = \frac{1}{2} x_a + \frac{1}{2} x_c$)