On the Passage of Gaussian Beams through Cusps in Ray Paths

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Abstract of the Dissertation

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Gaussian beams are asymptotic solutions to hyperbolic partial differentiable equations which propagate along null bicharacteristic curves in phase space. To build gaussian beams, one constructs a phase and an amplitude by using data along a specific null bicharacteristic. The current construction assumes that the ray path a beam follows in position space is smooth. In this work, we extend the construction to the case in which the ray path has cusps and deduce the phase shift that occurs when a beam passes through these cusps. We also present a new formula for the phase of a gaussian beam.
The dissertation of Neelesh Tiruviluamala is approved.

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2012
To my grandmother Jayam “Mumma” Sivaramakrishnan —
who brought her algebra book with her on her final trip to the hospital,
and who raised me, loved me, and shared with me her greatest secrets
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Vita

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CHAPTER 1

Introduction

Gaussian beams are asymptotic solutions to hyperbolic PDEs that are concentrated on the ray path projection of a single null bicharacteristic. We are interested in extending gaussian beam techniques to situations in which this ray path is not smooth. The equations involved in determining the amplitude and phase of a gaussian beam depend only on data along the null bicharacteristic in question. Solving the equations of geometric optics, on the other hand, involves working with data on a Lagrangian manifold that is comprised of all null bicharacteristics emanating from an initial surface.

1.1 A New Formula for the Phase of a Gaussian Beam

In what follows, we use $x = (x_0, x_1, \ldots, x_n) = (x_0, \tilde{x})$ and the same notation for $\xi, y$, and $\eta$. To construct a gaussian beam for the hyperbolic equation $Pv = 0$, one begins with the ansatz

$$u(x, k) = e^{ik\psi(x)} \left( a_0(x) + \frac{a_1(x)}{k} + \ldots + \frac{a_N(x)}{k^N} \right).$$

Letting $p_m(x, \xi)$ denote the principal symbol of $P$, recall that a null bicharacteristic curve $(x(s), \xi(s))$ is a solution to

$$\dot{x} = \frac{\partial p_m}{\partial \xi}(x, \xi), \quad \dot{\xi} = -\frac{\partial p_m}{\partial x}(x, \xi), \quad x(0) = y, \quad \xi(0) = \eta \neq 0,$$
where \( p_m(y, \eta) = 0 \) and thus \( p_m(x(s), \xi(s)) = 0 \) for all \( s \). We let \( \Gamma \) denote the curve traced out by \( x(s) \). Applying \( P \) to our ansatz yields

\[
P u = k^m p_m \left( x, \frac{\partial \psi}{\partial x} \right) a_0 e^{ik\psi} + O(k^{m-1}).
\]

The goal is to choose the amplitude \( a = a_0(x) + \frac{a_1(x)}{k} + \ldots + \frac{a_N(x)}{k^N} \) and the phase function \( \psi \) so that \( ||Pu|| \leq C_M k^{-M} \) where \( ||\cdot|| \) is an appropriate Sobolev norm [Ral82]. It is therefore reasonable to try and choose \( \psi(x) \) so that:

- \( \psi(x(s)) \) is real valued
- \( \text{Im} \left( \frac{\partial^2 \psi}{\partial x_1 \partial x_2}(x(s)) \right) \) is positive definite on vectors orthogonal to \( \dot{x}(s) \)
- \( p_m \left( x, \frac{\partial \psi}{\partial x} \right) \) vanishes to high order on \( \Gamma \)

Prescribing the first and second order partials of \( \psi \) along \( \Gamma \) is the most involved step in the gaussian beam construction. The remaining partials of \( \psi \) and those of each \( a_j \) along \( \Gamma \) can then be obtained by solving linear systems of ODEs.

In the case that \( p_m(x, \xi) \) is homogeneous in \( \xi \) of degree \( m \) (what follows will not be valid in the semiclassical setting), we have found a simple formula for \( \psi \) that has the correct first and second order partials along \( \Gamma \). For any specific null bicharacteristic determined by \( (\tilde{y}, \tilde{\eta}) \), we show that \( \psi(x, \tilde{y}, \tilde{\eta}) \) has the proper first and second order partials as prescribed by the gaussian beam construction. In other words, \( \psi(x, \tilde{y}, \tilde{\eta}) \) satisfies the three aforementioned requirements with \( p_m(x, \frac{\partial \psi}{\partial x}) \) specifically vanishing to order two on \( \Gamma(\tilde{y}, \tilde{\eta}) \). To find an \( f(x, \tilde{y}, \tilde{\eta}) \) that makes \( p_m(x, \frac{\partial \psi}{\partial x} + \frac{\partial f}{\partial x}) \) vanish to any specified order on \( \Gamma(\tilde{y}, \tilde{\eta}) \) then only involves solving linear systems of ODEs. This \( f \) can also be chosen so that its first and second order partials on \( \Gamma(\tilde{y}, \tilde{\eta}) \) vanish, so \( \psi + f \) will still satisfy the above three conditions.

### 1.2 This New Phase Formula is Global

Suppose that there is an isolated \( s_0 \) at which the velocity of the ray path \( \dot{x}(s, \tilde{y}, \tilde{\eta}) \) of the null bicharacteristic determined by \( (\tilde{y}, \tilde{\eta}) \) vanishes. We will refer to \( \dot{x}(s_0, \tilde{y}, \tilde{\eta}) \) as a cusp in
\( \Gamma(\tilde{y}, \tilde{\eta}) \). The current gaussian beam construction introduced by Ralston in [Ral82] does not provide a means to extend the beam past cusps. This is because the Hessian of the phase function on \( \Gamma \) satisfies a matrix Riccati equation whose solution blows up near cusps. The phase formula we describe in chapter 4 can be made sense of both before and after cusps.

1.3 Identifying the Phase Shift when a Beam Passes through a Cusp

The motivation for the above phase formula comes from the application of a microlocalized global Fourier integral operator that we construct in chapters 4 and 5. This operator is a parametrix for the Cauchy problem \( Pv = 0 \) with data specified at \( x_0 = 0 \). Using this parametrix, we also obtain an explicit formula for the jump in phase when a gaussian beam passes through a cusp.
CHAPTER 2

Geometric Optics and Gaussian Beams

2.1 Preliminaries

Gaussian beams are asymptotic solutions to linear hyperbolic partial differential equations. To motivate their definition, therefore, we will begin by reviewing some of the basic properties of hyperbolic PDEs.

Definition. Suppose that \( P \) is a linear partial differential operator of order \( m \) that acts on functions defined for \( x = (x_0, x_1, \ldots, x_n) = (x, \tilde{x}) \). The symbol \( p(x, \tilde{x}, \xi_0, \tilde{\xi}) \) of \( P \) is the function obtained by replacing \( \frac{\partial}{\partial x_j} \) in \( P \) with \( i\xi_j \) for \( 0 \leq j \leq n \). The principal symbol \( p_m \) is the order \( m \) component of \( p \). We similarly define \( p_k \) to be the order \( k \) component of \( p \) for \( 0 \leq j < m \). Another equivalent method of defining the principal symbol of \( P \) is by the equation

\[
p_m(x_0, \tilde{x}, \xi_0, \tilde{\xi}) = \lim_{\alpha \to \infty} \alpha^{-m} \left[ e^{-i\alpha(x_0 \cdot \xi_0 + \tilde{x} \cdot \tilde{\xi})} P \left( e^{i\alpha(x_0 \cdot \xi_0 + \tilde{x} \cdot \tilde{\xi})} \right) \right](x_0, \tilde{x}).
\]

This latter definition is easily generalized to pseudodifferential operators.

If we multiply \( e^{i\alpha[(x_0 - \tilde{x}_0) \cdot \xi_0 + (\tilde{x} - \tilde{\xi}) \cdot \tilde{\xi}]} \) by an appropriate smooth cutoff function centered at \((x_0, \tilde{x})\), we obtain a wave packet \( \psi \) localized to \((x_0, \tilde{x}, \alpha \xi_0, \alpha \tilde{\xi})\). To top order in \( \alpha \), then, \( \psi \) is roughly an eigenfunction of the linear partial differential operator \( P \) with eigenvalue \( \alpha^m p_m(x_0, \tilde{x}, \xi_0, \tilde{\xi}) \). This provides a heuristic interpretation of the principal symbol.

Definition. A linear partial differential operator \( P \) is hyperbolic if the polynomial \( f(s) = p_m(x_0, \tilde{x}, s, \tilde{\xi}) \) has \( m \) real roots when \( \tilde{\xi} \neq 0 \).
The following example is taken from [Esk11] and the explicit calculations that we omit below can be found therein.

**Example 2.1.** Suppose that $P$ is the second order linear differential operator given by

$$P u = -\frac{\partial^2 u}{\partial x_0^2} + \sum_{j,k=1}^{n} g^{jk}(x) \frac{\partial^2 u}{\partial x_j \partial x_k} + \sum_{j=1}^{n} b_j(x) \frac{\partial u}{\partial x_j} + b_0(x) \frac{\partial u}{\partial x_0} + c(x).$$

The principal symbol of $P$ is $p^2(x, \xi) = \xi_0^2 - \sum_{j,k=1}^{n} g^{jk}(x) \xi_j \xi_k$ and it can be factored as

$$p^2(x, \xi) = \left( \xi_0 - \sqrt{\sum_{j,k=1}^{n} g^{jk}(x) \xi_j \xi_k} \right) \left( \xi_0 + \sqrt{\sum_{j,k=1}^{n} g^{jk}(x) \xi_j \xi_k} \right).$$

Therefore, if the matrix $g$ whose $(j, k)$ entry is $g^{jk}$ is positive definite, then $P$ is strictly hyperbolic.

Before leaving this example, we will use it to motivate some upcoming definitions. If we were able to solve the Cauchy problem

$$Pu(x) = 0 \quad \text{for} \quad x_0 > 0$$

$$u(0, \tilde{x}) = A e^{ik\tilde{x} \cdot \tilde{\eta}}$$

$$\frac{\partial u}{\partial x_0}(0, \tilde{x}) = B e^{ik\tilde{x} \cdot \tilde{\eta}},$$

for $|\tilde{\eta}| = 1$, we would then be able to construct solutions to the general Cauchy problem by using Fourier synthesis. Using the method of geometric optics, one is not able to solve the above Cauchy problem exactly for each fixed $k$ and $\tilde{\eta}$, but one can create a family of approximate solutions $v(x, k, \tilde{\eta})$ whose accuracy increases as $k \to \infty$. More precisely, for any $M$, we can find $v$ such that

$$P v(x, k, \tilde{\eta}) = O \left( k^{-M} \right)$$

$$v(0, \tilde{x}, k, \tilde{\eta}) = A(k, \tilde{\eta}) e^{ik\tilde{x} \cdot \tilde{\eta}} + O \left( k^{-M} \right)$$

$$\frac{\partial v}{\partial x_0} (0, \tilde{x}, k, \tilde{\eta}) = B(k, \tilde{\eta}) e^{ik\tilde{x} \cdot \tilde{\eta}} + O \left( k^{-M} \right).$$
One can use this result to answer questions relating to solutions of hyperbolic PDEs with highly oscillatory initial data. For instance, to understand how the singularities in initial data propagate, one only needs to focus on the high frequency component of the data.

### 2.2 Geometric Optics

To build families of solutions like those mentioned in Example 2.1, one takes linear combinations of special asymptotic solutions which are of the form

$$
\begin{align*}
  u(x, k, \tilde{\eta}) &= e^{ik\phi(x, \tilde{\eta})} \left( a_0(x, \tilde{\eta}) + \frac{1}{k} a_1(x, \tilde{\eta}) + \ldots + \frac{1}{k^N} a_N(x, \tilde{\eta}) \right).
\end{align*}
$$

The above solution form is known as the geometric optics ansatz, or educated guess. In what follows, we do not require that $\tilde{\eta}$ be unit length. We call $\phi$ the phase and $a = a_0 + \frac{1}{k} a_1 + \ldots + \frac{1}{k^N} a_N$ the amplitude. Given a linear hyperbolic partial differential operator $P$ with principal symbol $p_m$, we simply apply $P$ to the above ansatz to deduce what the amplitude and phase must be for $Pu$ to vanish up to high order in $k$. We will throughout this article adopt the convention that for a function $w(z) : \mathcal{R}^m \to \mathcal{R}^n$, $\frac{\partial w}{\partial z}$ is the $n \times m$ matrix whose $(i,j)$ entry is $\frac{\partial w_i}{\partial z_j}$.

The result of applying $P$ to $u$ will be a complicated sum of terms that we can group together based on their $k^j$ term, where $j$ will range from $m$ down to $-N$. The easiest term to deduce is the $k^m$ term, which is the result of all of the highest order differentiations in $P$ falling on $e^{ik\phi(x, \tilde{\eta})}$:

$$
Pu = k^m p_m \left( x, \frac{\partial \phi}{\partial x} \right) e^{ik\phi} a_0 + \mathcal{O}(k^{m-1})
$$

As such, the first goal in geometric optics is to pick $\phi$ so that it solves the eikonal equation $p_m \left( x, \frac{\partial \phi}{\partial x} \right) = 0$. In fact, we want to solve the following Cauchy problem:
\[
p_m \left( x, \frac{\partial \phi}{\partial x}(x, \tilde{\eta}) \right) = 0, \quad \text{for} \quad x_0 > 0 \tag{2.2}
\]
\[
\phi(0, \tilde{x}, \tilde{\eta}) = \tilde{x} \cdot \tilde{\eta}
\]

Also, if we can pick \( \phi \) to solve equation (2.2), then each \( k^{m-j} \) term for \( j > 0 \) will only involve the unknowns \( a_0, a_1, \ldots, a_{j-1} \) since \( k^{m-j}p_m(x, \frac{\partial \phi}{\partial x})e^{ik\phi}a_j = 0 \). Thus, we can kill the \( k^{m-1} \) term by solving a first order linear PDE for \( a_0 \), we can then kill the \( k^{m-1} \) term by solving a first order linear PDE for \( a_1 \) (since \( a_0 \) is no longer an unknown), and we can continue until we have picked \( a_0, \ldots, a_N \) so that \( Pu = \mathcal{O}(k^{m-N-2}) \).

We will use the summation convention in everything that follows. It is straightforward, though not completely trivial, to work out that for \( i = 0, \ldots, N \), \( a_i \) must satisfy the first order linear PDE

\[
\frac{1}{i} \left( \partial p_m \left( x, \frac{\partial \phi}{\partial x} \right) \frac{\partial a_i}{\partial x_j} \right) + \left( \frac{1}{2i} \frac{\partial^2 p_m}{\partial \xi_j \partial \xi_k} \left( x, \frac{\partial \phi}{\partial x} \right) \frac{\partial^2 \phi}{\partial x_j \partial x_k} + p_{m-1} \left( x, \frac{\partial \phi}{\partial x} \right) \right) a_i = g_i(x), \tag{2.3}
\]

where \( g_i(x) \) is a complicated function of \( \phi, a_0, \ldots, a_{i-1} \) and their derivatives. Equations (2.2) and (2.3) have geometric interpretations for which we will need the following definitions to explain. From here on out, \( y = (y_0, \tilde{y}) \) and \( \eta = (\eta_0, \tilde{\eta}) \) will refer to \( n + 1 \) dimensional variables.

**Definition.** A bicharacteristic curve, or bicharacteristic, associated to an operator \( P \) with principal symbol \( p_m \) is a solution \((\hat{x}(s), \hat{\xi}(s))\) to the system of ordinary differential equations

\[
\dot{\hat{x}} = \frac{\partial p_m}{\partial \xi}(\hat{x}, \hat{\xi}), \quad \dot{\hat{\xi}} = -\frac{\partial p_m}{\partial x}(\hat{x}, \hat{\xi}), \quad (2.4)
\]

where the \( \cdot \) operation refers to differentiation with respect to the curve parameter \( s \). We call the \( x \)-space projection \( \hat{x}(s) \) of a bicharacteristic the ray path of that bicharacteristic. Note
that
\[ \hat{p}_m(\hat{x}(s), \hat{\xi}(s)) = \frac{\partial p_m}{\partial x_i}(\hat{x}(s), \hat{\xi}(s)) \hat{x}_i(s) + \frac{\partial p_m}{\partial \xi_i}(\hat{x}(s), \hat{\xi}(s)) \hat{\xi}_i(s) \]
\[ = -\hat{\xi}_i(s) \hat{x}_i(s) + \hat{x}_i(s) \hat{\xi}_i(s) = 0. \]

Therefore, if \( p_m(\hat{x}(0), \hat{\xi}(0)) = 0 \), then \( p_m(\hat{x}(s), \hat{\xi}(s)) = 0 \) for all \( s \) and such bicharacteristics \( (\hat{x}(s), \hat{\xi}(s)) \) are called null bicharacteristics.

**Definition.** Given an operator \( P \) with principal symbol \( p_m \), suppose that the real valued function \( \eta_0 \left( \tilde{y}, \frac{\tilde{\eta}}{||\tilde{\eta}||} \right) \) is such that \( p_m \left( 0, \tilde{y}, \eta_0 \left( \tilde{y}, \frac{\tilde{\eta}}{||\tilde{\eta}||} \right), \frac{\tilde{\eta}}{||\tilde{\eta}||} \right) = 0 \) for all \( \tilde{y} \) and \( \tilde{\eta} \) in \( \mathcal{R}^n \). Since \( p_m \) is homogeneous in \( \xi \) with degree \( m \), we have that
\[ 0 = ||\tilde{\eta}||^m p_m \left( 0, \tilde{y}, \eta_0 \left( \tilde{y}, \frac{\tilde{\eta}}{||\tilde{\eta}||} \right), \frac{\tilde{\eta}}{||\tilde{\eta}||} \right) = p_m \left( 0, \tilde{y}, ||\tilde{\eta}|| \eta_0 \left( \tilde{y}, \frac{\tilde{\eta}}{||\tilde{\eta}||} \right), \frac{\tilde{\eta}}{||\tilde{\eta}||} \right). \]

We therefore define \( \eta_0(\tilde{y}, \tilde{\eta}) := ||\tilde{\eta}|| \eta_0 \left( \tilde{y}, \frac{\tilde{\eta}}{||\tilde{\eta}||} \right) \) and so \( p_m \left( 0, \tilde{y}, \eta_0 (\tilde{y}, \tilde{\eta}), \tilde{\eta} \right) = 0 \) for all \( \tilde{y} \) and \( \tilde{\eta} \) in \( \mathcal{R}^n \). With the understanding that \( \eta_0(\tilde{y}, \tilde{\eta}) \) will be clear from context, we denote by \( (\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})) \) the null bicharacteristic curve whose initial \( (s = 0) \) value is
\[ (\hat{x}(0, \tilde{y}, \tilde{\eta}), \hat{\xi}(0, \tilde{y}, \tilde{\eta})) = (0, \tilde{y}, \eta_0(\tilde{y}, \tilde{\eta}), \tilde{\eta}). \]

We define \( \Lambda_{\tilde{\eta}} \) to be the \( n + 1 \) dimensional manifold that is the collection of bicharacteristics \( (\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})) \) for fixed \( \tilde{\eta} \neq 0 \) and we define \( \Lambda = \cup_{\tilde{\eta}} \Lambda_{\tilde{\eta}} \). We note that for all \( \lambda \neq 0 \),
\[ (\hat{x}(s, \tilde{y}, \lambda \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \lambda \tilde{\eta})) = (\hat{x}(\lambda^{m-1} s, \tilde{y}, \tilde{\eta}), \lambda \hat{\xi}(\lambda^{m-1} s, \tilde{y}, \tilde{\eta})) \]
since these two curves both satisfy equations (2.4) (here we use the homogeneity of \( p_m \) in \( \xi \) with the same initial data (here we use the fact that \( \eta_0 \) is homogeneous in \( \tilde{\eta} \) with degree 1). For those who are interested, this establishes the fact that \( \Lambda \) is a conic manifold. We only mention this fact to illustrate how the homogeneity of \( p_m \) and the conic nature of \( \Lambda \) are related.
Although we will provide direct proofs of our statements that do not rely on a knowledge of symplectic geometry, it is worth noting that $\Lambda_{\tilde{\eta}}$ is a Lagrangian submanifold of the cotangent bundle of $\mathcal{R}^{n+1}$. Therefore, if $\Lambda_{\tilde{\eta}}$ can be parameterized by $x$, then there is a function $\phi(x, \tilde{\eta})$ such that the graph of $\frac{\partial \phi}{\partial x}$ is $\Lambda_{\tilde{\eta}}$. The fact that $p_m$ is homogeneous implies that $\Lambda_{\tilde{\eta}}$ is conic in $\xi$ and this simplifies the expression for $\phi$. All of this is of interest to us because this $\phi$ is the solution to equation (2.2). In the following proposition, we will define $\phi$ and show that it satisfies (2.2). We will also establish some facts about $\phi$ that we will use later. Note that $\frac{\partial \tilde{x}_i}{\partial y_j}(0, \tilde{y}, \tilde{\eta}) = \delta_{ij}$ and that $\tilde{x}_0(0, \tilde{y}, \tilde{\eta}) = \frac{\partial p_m}{\partial \xi_0}(\tilde{x}(0, \tilde{y}, \tilde{\eta}), \tilde{\xi}(0, \tilde{y}, \tilde{\eta})) \neq 0$ by strict hyperbolicity. Thus, by the inverse function theorem, a neighborhood $\mathcal{U}$ of the $s = 0$ submanifold of $\Lambda$ can be parameterized by $(x, \tilde{\eta})$ as well as by $(s, \tilde{y}, \tilde{\eta})$. In what follows, we work exclusively in $\mathcal{U}$. We adopt the convention that for any function $h(x, \tilde{\eta})$ defined on $\mathcal{U}$, $\dot{h}(s, \tilde{y}, \tilde{\eta}) = h(\dot{x}(s, \tilde{y}, \tilde{\eta}), \dot{\tilde{\eta}})$ and that for any function $\dot{g}(s, \tilde{y}, \tilde{\eta})$ defined on $\mathcal{U}$, $\dot{g}(s, \tilde{y}, \tilde{\eta}) = \dot{g}(s, \tilde{y}, \tilde{\eta})$. Before we define $\phi$, we will prove a lemma that relies on the fact that $p_m$ is homogeneous in $\xi$.

**Lemma 2.2.** The following relations hold:

\[
\begin{align*}
\dot{\hat{x}}_i(s, \tilde{y}, \tilde{\eta}) \frac{\partial}{\partial y_j} \left[ \frac{\partial p_m}{\partial \xi_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})) \right] &= -\frac{\partial \hat{\xi}_i(s, \tilde{y}, \tilde{\eta})}{\partial \hat{y}_j} \frac{\partial p_m}{\partial \hat{\xi}_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})), \\
\dot{\hat{\xi}}_i(s, \tilde{y}, \tilde{\eta}) \frac{\partial}{\partial \eta_j} \left[ \frac{\partial p_m}{\partial \xi_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})) \right] &= -\frac{\partial \hat{\xi}_i(s, \tilde{y}, \tilde{\eta})}{\partial \hat{\eta}_j} \frac{\partial p_m}{\partial \hat{\xi}_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})).
\end{align*}
\]

**Proof.** Since $p_m(x, \xi)$ is homogeneous of degree $m$ in $\xi$, Euler’s homogeneous function theorem yields

\[\xi_i \frac{\partial p_m}{\partial \xi_i} = mp_m\]

and so $\dot{\xi}_i(s, \tilde{y}, \tilde{\eta}) \frac{\partial p_m}{\partial \xi_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})) = 0$. Differentiating this equation in $y_j$ and $\eta_j$ respectively yields the above relations. \qed

**Proposition 2.3.** Define $\dot{\phi}(s, \tilde{y}, \tilde{\eta}) = y_j \eta_j$. The following relations hold for $\phi(x, \tilde{\eta}) = y_j(x, \tilde{\eta}) \eta_j$:

\[
\begin{align*}
\frac{\partial \phi}{\partial x_i}(x, \tilde{\eta}) &= \xi_i(x, \tilde{\eta}), \\
\frac{\partial \phi}{\partial \eta_j}(x, \tilde{\eta}) &= y_j(x, \tilde{\eta}),
\end{align*}
\]

for $i = 0, \ldots, n$ and $j = 1, \ldots, n$. 9
Proof. We start by differentiating $\hat{\xi}_i(s, \tilde{y}, \tilde{\eta}) \frac{\partial \hat{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta})$ in $s$ and use Lemma 2.2 to obtain

$$
\frac{\partial}{\partial s} \left[ \hat{\xi}_i(s, \tilde{y}, \tilde{\eta}) \frac{\partial \hat{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}) \right] = \hat{\xi}_i(s, \tilde{y}, \tilde{\eta}) \frac{\partial \hat{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}) + \hat{\xi}_i(s, \tilde{y}, \tilde{\eta}) \frac{\partial \hat{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta})
$$

$$
= - \frac{\partial p_m}{\partial x_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})) \frac{\partial \hat{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}) + \hat{\xi}_i(s, \tilde{y}, \tilde{\eta}) \partial \left[ \frac{\partial p_m}{\partial \xi_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})) \right]
$$

$$
= - \frac{\partial p_m}{\partial x_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})) \frac{\partial \hat{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}) - \frac{\partial \hat{\xi}_i(s, \tilde{y}, \tilde{\eta})}{\partial y_j} \frac{\partial p_m}{\partial \xi_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta}))
$$

$$
= - \frac{\partial p_m}{\partial x_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})) \frac{\partial \hat{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}) = 0.
$$

Therefore, $\hat{\xi}_i(s, \tilde{y}, \tilde{\eta}) \frac{\partial \hat{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}) = \hat{\xi}_i(0, \tilde{y}, \tilde{\eta}) \frac{\partial \hat{x}_i}{\partial y_j}(0, \tilde{y}, \tilde{\eta}) = \eta_j$.

Finally, since $\frac{\partial \hat{\phi}}{\partial y_j} = \eta_j$ and $\hat{\phi}(s, \tilde{y}, \tilde{\eta}) = \phi(\hat{x}(s, \tilde{y}, \tilde{\eta}), \eta)$, it follows that

$$
\frac{\partial \hat{\phi}}{\partial x_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})) \frac{\partial \hat{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}) = \eta_j
$$

and thus

$$
\hat{\xi}_i(s, \tilde{y}, \tilde{\eta}) \frac{\partial \hat{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}) = \frac{\partial \hat{\phi}}{\partial x_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})) \frac{\partial \hat{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}).
$$

Also, note that

$$
\hat{\xi}_i(s, \tilde{y}, \tilde{\eta}) \frac{\partial \hat{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}) = \hat{\xi}_i(s, \tilde{y}, \tilde{\eta}) \frac{\partial p_m}{\partial \xi_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta}))
$$

$$
= m \frac{\partial p_m}{\partial \xi_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta}))
$$

$$
= 0 = \frac{\partial \hat{\phi}}{\partial s}
$$

$$
= \frac{\partial \hat{\phi}}{\partial x_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})) \frac{\partial \hat{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}).
$$

Since $\frac{\partial \hat{x}(s, \tilde{y}, \tilde{\eta})}{\partial (s, \tilde{y}, \tilde{\eta})}$ is invertible, it follows that $\hat{\xi}_i(s, \tilde{y}, \tilde{\eta}) = \frac{\partial \hat{\phi}}{\partial x_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \eta)$ and hence by changing variables that $\xi_i(x, \eta_\tilde{\eta}) = \frac{\partial \hat{\phi}}{\partial x_i}(x, \eta)$.  

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Using Lemma 2.2 again, we obtain by using an identical argument to the one carried out above that
\[
\frac{\partial}{\partial s} \left[ \hat{\xi}_i(s, \tilde{y}, \tilde{\eta}) \frac{\partial \hat{x}_i}{\partial \eta_j}(s, \tilde{y}, \tilde{\eta}) \right] = 0.
\]
Thus, \( \hat{\xi}_i(s, \tilde{y}, \tilde{\eta}) \frac{\partial \hat{x}_i}{\partial \eta_j}(s, \tilde{y}, \tilde{\eta}) = \hat{\xi}_i(0, \tilde{y}, \tilde{\eta}) \frac{\partial \hat{x}_i}{\partial \eta_j}(0, \tilde{y}, \tilde{\eta}) = 0 \) and so
\[
\frac{\partial \phi}{\partial x_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \tilde{\eta}) \frac{\partial \hat{x}_i}{\partial \eta_j}(s, \tilde{y}, \tilde{\eta}) = 0.
\]
By definition, we have that \( \frac{\partial \hat{\phi}}{\partial \eta_j}(s, \tilde{y}, \tilde{\eta}) = y_j \) and that \( \hat{\phi}(s, \tilde{y}, \tilde{\eta}) = \phi(\hat{x}(s, \tilde{y}, \tilde{\eta}), \tilde{\eta}) \), so
\[
y_j = \frac{\partial \hat{\phi}}{\partial \eta_j}(s, \tilde{y}, \tilde{\eta})
= \frac{\partial \phi}{\partial x_i}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \tilde{\eta}) \frac{\partial \hat{x}_i}{\partial \eta_j}(s, \tilde{y}, \tilde{\eta}) + \frac{\partial \phi}{\partial \eta_j}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \tilde{\eta})
= \frac{\partial \phi}{\partial \eta_j}(\hat{x}(s, \tilde{y}, \tilde{\eta}), \tilde{\eta}).
\]
Changing variables yields \( \frac{\partial \phi}{\partial \eta_j}(x, \tilde{\eta}) = y_j(x, \tilde{\eta}). \)

**Corollary 2.4.** The function \( \phi(x, \tilde{\eta}) \) defined in proposition 2.3 solves the Cauchy problem (2.2) for the eikonal equation.

**Proof.** We see that \( p_m \left( x, \frac{\partial \phi}{\partial x}(x, \tilde{\eta}) \right) = p_m \left( x, \xi(x, \tilde{\eta}) \right) = 0 \) and that \( \phi(0, \tilde{x}, \tilde{\eta}) = y_j(0, \tilde{x}, \tilde{\eta}) \eta_j = \tilde{x} \cdot \tilde{\eta}. \)

**Corollary 2.5.** The function \( \phi(x, \tilde{\eta}) \) defined in proposition 2.3 is homogeneous of degree 1 in \( \tilde{\eta} \).

**Proof.** This follows from Euler’s homogeneous function theorem since
\[
\phi(x, \tilde{\eta}) = \eta_j \frac{\partial \phi}{\partial \eta_j}(x, \tilde{\eta}).
\]
Thus, we have found a function \( \phi(x, \tilde{\eta}) \) such that the graph of \( \frac{\partial \phi}{\partial x} \) is \( \Lambda_{\tilde{\eta}} \) near \( s = 0 \) and verified in Corollary 2.4 that this \( \phi \) satisfies the Cauchy problem (2.2). It is also at this point possible to provide a geometric context for equations (2.3). Since

\[
\frac{\partial p_m}{\partial \xi_j} \left( x, \frac{\partial \phi}{\partial x}(x, \tilde{\eta}) \right) \frac{\partial a_i}{\partial x_j}(x, \tilde{\eta}) = \frac{\partial p_m}{\partial \xi_j}(x, \xi(x, \tilde{\eta})) \frac{\partial a_i}{\partial x_j}(x, \tilde{\eta})
\]

\[
= \dot{x}_j(s(x, \tilde{\eta}), \tilde{y}(x, \tilde{\eta}), \tilde{\eta}) \frac{\partial a_i}{\partial x_j}(x, \tilde{\eta})
\]

\[
= \dot{a}_i(s(x, \tilde{\eta}), \tilde{y}(x, \tilde{\eta}), \tilde{\eta}),
\]

it is evident that the differential operator in equations (2.3) involves differentiation along the ray paths of the null bicharacteristics of \( p_m \).

For a strictly hyperbolic operator \( P \) with principal symbol \( p_m \), there are \( m \) choices for the function \( \eta_0(\tilde{y}, \tilde{\eta}) \) used in the definition of \( \Lambda \), corresponding to the \( m \) distinct roots of \( f(s) = p_m(0, \tilde{y}, s, \tilde{\eta}) \). Therefore, there are \( m \) different solutions \( \phi_1, \ldots, \phi_m \) to the Cauchy problem (2.2), each with its own corresponding amplitude. It is then straightforward to use linear combinations of \( m \) functions in the form of the geometric optics ansatz (2.1) to build families of solutions like those mentioned in Example 2.1.

### 2.3 Gaussian Beams

The method of geometric optics produces highly oscillatory asymptotic solutions to a strictly hyperbolic equation \( P v = 0 \). The method of gaussian beams starts off with the geometric optics ansatz and also produces highly oscillatory asymptotic solutions. The difference is that a gaussian beam solution, or simply gaussian beam, becomes concentrated on the ray path of a single null bicharacteristic. As such, gaussian beams are robust because their construction relies on data along a single null bicharacteristic instead of the family of null bicharacteristics \( \Lambda_{\tilde{\eta}} \). Furthermore, we will see that a gaussian beam can be built along the entire ray path.

Again, we know by strict hyperbolicity that there are \( m \) continuous functions \( \eta_0^1(\tilde{y}, \tilde{\eta}), \ldots, \eta_0^m(\tilde{y}, \tilde{\eta}) \),
\[ \eta_0^m (\tilde{y}, \tilde{\eta}) \) such that \( p_m (0, \tilde{y}, \eta_0^i (\tilde{y}, \tilde{\eta})) = 0 \) for \( i = 1, \ldots, m \). We also have seen that these \( \eta_0^i \) are homogeneous in \( \tilde{\eta} \) of degree 1. As we did previously, we will drop the superscript, work simply with one \( \eta_0 \), and we will use all the same notation as that introduced in the geometric optics section. For fixed \((\tilde{y}, \tilde{\eta})\), we will outline how to construct a gaussian beam concentrated on the ray path of the null bicharacteristic

\[ (x(s), \xi(s)) = (\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})) \]

The idea for the following construction comes from [Ral82]. We begin with the geometric optics ansatz

\[ u(x, k) = e^{ik\psi(x)} \left( a_0(x) + \frac{1}{k} a_1(x) + \ldots + \frac{1}{k^N} a_N(x) \right) \tag{2.5} \]

and suppress the \((\tilde{y}, \tilde{\eta})\) dependence of the functions involved. We will choose \( \psi \) so that \( \psi (x(s)) \) is real-valued, but we no longer require that \( \psi \) be real valued everywhere. In fact, we will choose \( \psi \) so that \( \text{Im} \frac{\partial^2 \psi}{\partial x_i \partial x_j} (x(s)) \) is positive definite on vectors orthogonal to \( \dot{x}(s) \). This will lead \( |e^{ik\psi}| \) to decay like a gaussian distribution with variance proportional to \( k^{-1} \) on planes perpendicular to \( x(s) \). Our goal, as it was with the geometric optics construction, is to pick the amplitude and phase so that \( Pu \) is \( O \left( k^{-M} \right) \) for \( M \) large. The result of applying \( P \) to \( u \) will be a complicated sum of terms that we can group together based on their \( k^j \) term, where \( j \) will range from \( m \) down to \(-N\). The \( k^m \) term, which is the result of all of the highest order differentiations in \( P \) falling on \( e^{ik\psi(x)} \), will be

\[ Pu = k^m p_m \left( x, \frac{\partial \psi}{\partial x} \right) e^{ik\psi} a_0 + O \left( k^{m-1} \right) . \]

Because we can rely on the gaussian decay of \( e^{ik\psi} \) away from \( x(s) \), we no longer need \( p_m \left( x, \frac{\partial \psi}{\partial x} \right) = 0 \), but instead, only that

\[ f(x) := p_m \left( x, \frac{\partial \psi}{\partial x}(x) \right) \]

vanishes to high order along \( x(s) \). More precisely, using elementary calculus it is possible to show that if all partial derivatives of \( f \) up to and including order \( r-1 \) vanish along \( x(s) \),

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then

\[ p_m \left( x, \frac{\partial \psi}{\partial x}(x) \right) e^{ik\psi} a_0 = \mathcal{O}(k^{-r/2}) \]

if \( \text{Im} \frac{\partial^2 \psi}{\partial x \partial x_j}(x(s)) \) has been chosen as described above. Therefore, our first goal will be to prescribe the partial derivatives of \( \psi \) along \( x(s) \) so that \( f \) and its lower order partial derivatives vanish along \( x(s) \). The equations for \( f(x(s)) = 0, \frac{\partial f}{\partial x_j}(x(s)) = 0, \) and \( \frac{\partial^2 f}{\partial x_i \partial x_j}(x(s)) = 0 \) are

\[ 0 = p_m, \quad (2.6) \]

\[ 0 = \frac{\partial p_m}{\partial x_j} + \frac{\partial p_m}{\partial \xi_k} \frac{\partial^2 \psi}{\partial x_j \partial x_k}, \quad (2.7) \]

\[ 0 = \frac{\partial^2 p_m}{\partial x_i \partial x_j} + \frac{\partial^2 p_m}{\partial \xi_k \partial x_j} \frac{\partial^2 \psi}{\partial x_i \partial x_k} + \frac{\partial^2 p_m}{\partial \xi_i \partial \xi_k} \frac{\partial^2 \psi}{\partial x_i \partial x_k} + \frac{\partial^2 p_m}{\partial x_i \partial \xi_k} \frac{\partial^2 \psi}{\partial x_j \partial x_k} + \frac{\partial p_m}{\partial \xi_k} \frac{\partial^3 \psi}{\partial x_i \partial x_j \partial x_k}, \quad (2.8) \]

By setting \( \frac{\partial \psi}{\partial x}(x(s)) = \xi(s) \), we satisfy equation (2.6). Our next goal is to prescribe the matrix

\[ M_{ij}(s) := \frac{\partial^2 \psi}{\partial x_i \partial x_j}(x(s)) \]

of second order partials of \( \psi \) along \( x(s) \). Differentiating \( \frac{\partial \psi}{\partial x_j}(x(s)) = \xi_j(s) \) in \( s \), we see that \( M(s) \) must satisfy

\[ M_{kj}(s) \dot{\xi}_k(s) = \frac{\partial^2 \psi}{\partial x_k \partial x_j}(x(s)) \dot{\xi}_j(s) = \dot{\xi}_j(s) \]

\[ (2.10) \]

for \( j = 0, \ldots, n \). Because \( (x(s), \xi(s)) \) is a bicharacteristic, this equation is identical to equation (2.7) once we replace \( \frac{\partial \psi}{\partial x}(x(s)) \) with \( \xi(s) \). Similarly, differentiating equation (2.9) in \( s \) reveals that the third order partials of \( \psi \) along \( x(s) \) must satisfy the compatibility relations

\[ M_{ij}(s) = \frac{\partial p_m}{\partial \xi_k}(x(s)) \frac{\partial \psi}{\partial x_i \partial x_j \partial x_k}(x(s)). \quad (2.11) \]

We will assume that equation (2.11) holds and worry about insuring that it indeed does later...
when prescribing third order partials. After defining the matrices $A(s), B(s),$ and $C(s)$ as

$$A_{ij}(s) = \frac{\partial^2 p_m}{\partial x_i \partial x_j}(x(s), \xi(s)),$$
$$B_{ij}(s) = \frac{\partial^2 p_m}{\partial x_i \partial \xi_j}(x(s), \xi(s)),$$
$$C_{ij}(s) = \frac{\partial^2 p_m}{\partial \xi_i \partial \xi_j}(x(s), \xi(s)),$$

we can consolidate the equations (2.8) for all $i$ and $j$ into the matrix Riccati equation

$$0 = A + MB^T + BM + MCM + \dot{M}. \quad (2.12)$$

For the gaussian beam construction to work, therefore, $M$ must satisfy equation (2.10) and equation (2.12), $\text{Im}[M(s)]$ must be positive definite on vectors orthogonal to $\dot{x}(s)$, and $M(s)$ must be symmetric. It turns out, somewhat surprisingly, that even though equation (2.12) is not linear, for every appropriate choice of $M(0)$, there is a unique $M(s)$ that satisfies all of the above conditions for all $s > 0$. By appropriate choice, we mean that $\text{Im}[M(0)]$ must be positive definite on vectors orthogonal to $\dot{x}(0)$, $M(0)$ must be symmetric, and that $M_{kj}(0) \dot{x}_k(0) = \dot{\xi}_j(0)$.

We will now describe how $M(s)$ is determined for all $s > 0$ once $M(0)$ has been chosen as prescribed above. Since $P$ is strictly hyperbolic, its principal symbol $p_m$ when viewed as a polynomial in $\xi_0$ shares no roots with $\dot{x}_0 = \frac{\partial p_m}{\partial \xi_0}$. Thus, $\dot{x}_0(0) \neq 0$ and so $\{\dot{x}(0), e_1, e_2, \ldots, e_n\}$ is a basis for $\mathcal{R}^{n+1}$, where $e_j$ denotes the $j$th standard basis vector. For any two $n + 1$ dimensional vectors $v$ and $w$, we will use the notation $v | w$ to denote the $2n + 2$ dimensional vector obtained by appending vector $w$ to $v$. For each $s$, the flow associated with $p_m$ induces a map from the tangent space of $\mathcal{R}^{2n+2}$ at $(x(0), \xi(0))$ to the tangent space of $\mathcal{R}^{2n+2}$ at $(x(s), \xi(s))$. Noting that $M(0) \dot{x}(0) = \dot{\xi}(0)$, we have that the vectors

$$\left\{ \dot{x}(0) | \dot{\xi}(0), e_1 | (M(0)e_1), e_2 | (M(0)e_2), \ldots, e_n | (M(0)e_n) \right\} \quad (2.13)$$

span an $n + 1$ dimensional subspace of $\mathcal{C}^{2n+2}$. Considering the real parts and imaginary parts of these vectors separately as elements of the tangent space of $\mathcal{R}^{2n+2}$ at $(x(0), \xi(0))$ and then
by using linearity, \( p_m \) also induces a map from these vectors to vectors which we will denote by

\[
\{ \delta x_0(s)|\delta \xi_0(s), \delta x_1(s)|\delta \xi_1(s), \delta x_2(s)|\delta \xi_2(s), \ldots, \delta x_n(s)|\delta \xi_n(s) \}.
\]

None of the above commentary is necessary to define \( M(s) \), but it provides a helpful geometrical interpretation for \( \delta x_i(s)|\delta \xi_i(s) \) in what follows. We can more explicitly define \( \delta x_i(s)|\delta \xi_i(s) \) for \( i = 0, \ldots, n \) by solving the linear system

\[
\dot{x}_i = B^T (\delta x_i) + C (\delta \xi_i),
\]

\[
\dot{\xi}_i = -A (\delta x_i) - B (\delta \xi_i).
\]

We denote by \( v^i \) the \( i \)th component of vector \( v \). If \( \delta x_j(0)|\delta \xi_j(0) \) is defined to be the \( j \)th vector in (2.13), we claim that these initial conditions produce the following solutions to the linear system (2.14):

\[
\delta x_0(s) = \dot{x}(s),
\]

\[
\delta \xi_0(s) = \dot{\xi}(s),
\]

\[
\delta x_j(s) = \frac{\partial \dot{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}) + \frac{\partial \dot{\xi}_i}{\partial \eta_k}(s, \tilde{y}, \tilde{\eta})M_{kj}(0) \quad \text{for} \quad j = 1, \ldots, n,
\]

\[
\delta \xi_j(s) = \frac{\partial \dot{\xi}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}) + \frac{\partial \dot{x}_i}{\partial \eta_k}(s, \tilde{y}, \tilde{\eta})M_{kj}(0) \quad \text{for} \quad j = 1, \ldots, n.
\]

First, it is clear that each vector \( \frac{\partial \dot{x}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}), \frac{\partial \dot{x}_i}{\partial \eta_k}(s, \tilde{y}, \tilde{\eta}), \frac{\partial \dot{\xi}_i}{\partial y_j}(s, \tilde{y}, \tilde{\eta}), \frac{\partial \dot{\xi}_i}{\partial \eta_k}(s, \tilde{y}, \tilde{\eta}) \), and \( \dot{x}(s)|\dot{\xi}(s) \) individually solves the system (2.14) for all \( j = 1, \ldots, n \). Therefore, the above linear combinations of these vectors are also solutions to this system. By definition, \( \delta x_0(0)|\delta \xi_0(0) = \dot{x}(0)|\dot{\xi}(0) \) so it only remains to check that \( \delta x_j(0)|\delta \xi_j(0) = e_j|(M(0)e_j) \) for \( j = 1, \ldots, n \). Since \( \dot{x}_0(0, \tilde{y}, \tilde{\eta}) = 0 \) and \( \dot{x}_i(0, \tilde{y}, \tilde{\eta}) = y_i \) for \( i = 1, \ldots, n \), it follows that \( \delta x_j(0) = e_j \). Similarly, from the fact that \( \dot{\xi}_i(0, \tilde{y}, \tilde{\eta}) = \eta_i \) for \( i = 1, \ldots, n \) it follows that \( \delta \xi_j(0) = M_{ij}(0) \) for \( i \) and \( j \) between 1 and \( n \). Since \( \dot{\xi}_0(0, \tilde{y}, \tilde{\eta}) = \eta_0(\tilde{y}, \tilde{\eta}) \), the final fact that we have to check is that

\[
M_{0j}(0) = \frac{\partial \eta_0}{\partial y_j}(\tilde{y}, \tilde{\eta}) + \frac{\partial \eta_0}{\partial \eta_k}(\tilde{y}, \tilde{\eta})M_{kj}(0)
\]
for \(j = 1, \ldots, n\). By differentiating \(p_m(0, \tilde{y}, \eta_0(\tilde{y}, \tilde{\eta}), \tilde{\eta}) = 0\) in \(y_j\) and \(\eta_k\) respectively and evaluating the results at \((\tilde{y}, \tilde{\eta})\), we obtain

\[
0 = \frac{\partial p_m}{\partial x_j} + \frac{\partial p_m}{\partial \xi_0} \frac{\partial \eta_0}{\partial y_j}
\]

\[
0 = \frac{\partial p_m}{\partial \xi_k} + \frac{\partial p_m}{\partial \xi_0} \frac{\partial \eta_0}{\partial \eta_k}
\]

Since \(\dot{\xi}_0 \neq 0\), it follows that we can rewrite equation (2.15) as

\[
\dot{x}_0 M_0j(0) = \dot{\xi}_j - \dot{x}_k M_{kj}(0),
\]

where we keep in mind that the index \(k\) is restricted to be between 1 and \(n\). Using the fact that \(M(0)\) is symmetric, the above equations for \(j = 1, \ldots, n\) are equivalent to the fact that \(M(0)\dot{x}(0) = \dot{\xi}(0)\) and so we are done.

Without getting too bogged down in all the computations, the important point is that the vector \(\delta x_j(s)\mid \delta \xi_j(s)\) has a simple geometric interpretation and that it can be written as a natural linear combination of the derivatives of \((\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta}))\). Once this is done, \(M(s)\) can be explicitly defined as the matrix such that for all \(i = 0, \ldots, n\),

\[
M(s) (\delta x_i(s)) = \delta \xi_i(s).
\]

The details concerning why this \(M(s)\) meets all the necessary conditions can be found in [Ral82]. Again, what is most surprising is that \(M(s)\) is well defined for all \(s > 0\). Consulting [Ral82] will help the reader understand that this is a consequence of the fact that \(\dot{x}(s) \neq 0\) for all \(s\) due to strict hyperbolicity and the fact that \(\text{Im}[M(0)]\) is chosen to be positive definite on the orthogonal complement of \(\dot{x}(0)\). We have thus described how to prescribe the first and second order partials of \(\psi\) along \(x(s)\) for all \(s > 0\).

Prescribing the higher order derivatives is much easier. Although the exact expressions are long, it is easy to see without any work that the equations \(\frac{\partial^3 f}{\partial x_l \partial x_i \partial x_j}(x(s)) = 0\) become a linear system of ordinary differential equations for the third order partials of \(\psi\) along \(x(s)\) once we impose the compatibility relation

\[
\frac{\partial}{\partial s} \left[ \frac{\partial^3 \phi}{\partial x_l \partial x_i \partial x_j}(x(s)) \right] = \frac{\partial p_m}{\partial \xi_k}(x(s)) \frac{\partial^4 \psi}{\partial x_l \partial x_i \partial x_j \partial x_k}(x(s)).
\]

(2.17)
It might initially be alarming that we also have to make sure that the third order partials satisfy the compatibility relation (2.11), but this is ultimately not a problem. Indeed, if we choose the third order partials to be compatible at $x(0)$, it turns out that the solutions to the linear system with this initial data will stay compatible for all $s$. Thus, we can prescribe the third order partials of $\psi$ along $x(s)$ for all $s > 0$ as well. This same reasoning works for all higher order partials and so we are able to define $\psi$ so that $f$ vanishes to any prescribed order along $x(s)$. We will not go through the details here, but prescribing the appropriate partials of $a_0, \ldots, a_N$ along $x(s)$ similarly only involves solving linear systems of ODEs.
CHAPTER 3

The Tricomi Example

The standard construction for a gaussian beam centered around a null bicharacteristic \((x(s),\xi(s))\) relies on the fact that \(\dot{x}(s) \neq 0\) for all \(s > 0\). We will work through an informative example in which the null bicharacteristic does not exhibit this property. This will motivate the development of the necessary machinery to tackle the general problem of building gaussian beams centered on null bicharacteristics with cusps.

3.1 Generalized Geometric Optics

We define the operator \(P\) acting on functions of two variables as

\[
P = \frac{\partial^2}{\partial x_0^2} + x_0 \frac{\partial^2}{\partial x_1^2}.
\]

(3.1)

\(Pu = 0\) is commonly called the Tricomi equation and we call \(P\) the Tricomi operator. The symbol of \(P\) is \(p_2 = -\xi_0^2 - x_0\xi_1^2\). We see that \(f(s) := p_2(x_0, x_1, s, \xi_1)\) has two distinct real roots if and only if \(x_0 < 0\). In other words, \(P\) changes type (from hyperbolic to elliptic) across the line \(x_0 = 0\). We are going to focus on the null bicharacteristic of \(P\) given by

\[
(x_0(s), x_1(s), \xi_0(s), \xi_1(s)) = \left(-s^2, \frac{2}{3}s^3, s, 1\right).
\]

(3.2)

Note that the ray path \((x_0(s), x_1(s))\) of this null bicharacteristic passes through \((-1, -\frac{2}{3})\) at \(s = -1\), \((0, 0)\) at \(s = 0\), and \((-1, \frac{2}{3})\) at \(s = 1\). There is a cusp in the ray path at \((0, 0)\) in the sense that \((\dot{x}_0(0), \dot{x}_1(0)) = 0\). This can also be seen by graphing the ray path on the
plane using the relation
\[ x_0 = -\left(\frac{3x_1}{2}\right)^\frac{2}{3}. \]

The standard gaussian beam construction, therefore, breaks down at \( s = 0 \) and we are interested in seeing how we can extend beams through cusps like this one.

Every null bicharacteristic whose ray path passes through \((0, y_1)\) at \( s = 0 \) is of the form
\[
(\hat{x}_0(s, y_1, \eta_1), \hat{x}_1(s, y_1, \eta_1), \hat{\xi}_0(s, y_1, \eta_1), \hat{\xi}_1(s, y_1, \eta_1)) = \left( -\eta_1^2 s^2, \frac{2}{3} \eta_1^3 s^3 + y_1, \eta_1^2 s, \eta_1 \right). \tag{3.3}
\]

Using the notation from section 2.2, we let \( \Lambda \) denote the collection of all of these null bicharacteristics and \( \Lambda_{\eta_1} \) denote the collection of null bicharacteristics for fixed \( \eta_1 \). In this instance, \( \Lambda_{\eta_1} \) cannot be parameterized by \((x_0, x_1)\) near the submanifold of \( \Lambda_{\eta_1} \) defined by \( s = 0 \). We can therefore not find a solution \( \phi \) to equation (2.2) which is a function of \((x_0, x_1, \eta_1)\) in the way we did in section 2.2. We can, however, parameterize \( \Lambda_{\eta_1} \) globally by \((x_1, \xi_0)\) and so our first goal (we will see exactly why in a bit) will be to find a function \( \hat{S}(x_1, \xi_0, \eta_1) \) that satisfies
\[
p_2 \left( -\frac{\partial \hat{S}}{\partial \xi_0}(x_1, \xi_0, \eta_1), x_1, \xi_0, \frac{\partial \hat{S}}{\partial x_1}(x_1, \xi_0, \eta_1) \right) = 0, \tag{3.4}
\]
\[ \hat{S}(y_1, \hat{\xi}_0(0, y_1, \eta_1), \eta_1) = y_1 \eta_1 - \hat{x}_0(0, y_1, \eta_1) \hat{\xi}_0(0, y_1, \eta_1). \]

In this example, \( \hat{\xi}_0(0, y_1, \eta_1) = 0 \). We define \( \hat{s}(x_1, \xi_0, \eta_1) \) and \( \hat{y}_1(x_1, \xi_0, \eta_1) \) to be the solutions of the system
\[
x_1 = \frac{2}{3} \eta_1^3 s^3 + \hat{y}_1,
\]
\[ \xi_0 = \eta_1^2 \hat{s}. \]

We will adopt the convention that for a function \( \hat{h}(s, y_1, \eta_1) \),
\[ \hat{h}(x_1, \xi_0, \eta_1) = \hat{h}(\hat{s}(x_1, \xi_0, \eta_1), \hat{y}_1(x_1, \xi_0, \eta_1), \eta_1). \]

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Using arguments similar to those used in the proof of proposition 2.3, we can show that
\[ \tilde{S}(x_1, \xi_0, \eta_1) = \tilde{y}_1(x_1, \xi_0, \eta_1)\eta_1 - \tilde{x}_0(x_1, \xi_0, \eta_1)\xi_0 \]
satisfies
\[ \frac{\partial \tilde{S}}{\partial x_1}(x_1, \xi_0, \eta_1) = \tilde{\xi}_1(x_1, \xi_0, \eta_1), \]
\[ \frac{\partial \tilde{S}}{\partial \xi_0}(x_1, \xi_0, \eta_1) = -\tilde{x}_0(x_1, \xi_0, \eta_1), \] (3.5)
\[ \frac{\partial \tilde{S}}{\partial \eta_1}(x_1, \xi_0, \eta_1) = \tilde{y}_1(x_1, \xi_0, \eta_1). \]

Rather than verifying the above relations in this manner, however, it is easy instead to compute the following explicit formulas for \( \tilde{y}_1, \tilde{x}_0, \tilde{\xi}_1, \) and \( \tilde{S} \):
\[ \tilde{y}_1(x_1, \xi_0, \eta_1) = x_1 - \frac{2\xi_0^3}{3\eta_1^3}, \]
\[ \tilde{x}_0(x_1, \xi_0, \eta_1) = -\frac{\xi_0^2}{\eta_1^2}, \]
\[ \tilde{\xi}_1(x_1, \xi_0, \eta_1) = \eta_1, \]
\[ \tilde{S}(x_1, \xi_0, \eta_1) = x_1\eta_1 + \frac{\xi_0^3}{3\eta_1^2}. \]

It is then clear that equations (3.5) hold and therefore that \( \tilde{S} \) satisfies (3.4).

Using \( \tilde{S} \), it is possible to build asymptotic (\( |\eta_1| \to \infty \)) solutions to \( Pv = 0 \) of the form
\[ \tilde{v}(x_0, x_1, \eta_1) = \left( \frac{|\eta_1|}{2\pi} \right)^{\frac{3}{2}} \int_\mathcal{R} \tilde{a}(x_1, \xi_0, \eta_1) e^{i|\eta_1|(\tilde{S}(x_1, \xi_0, \eta_1) + x_0\xi_0)} d\xi_0, \] (3.6)

where
\[ \tilde{a}(x_1, \xi_0, \eta_1) = \tilde{a}_0(x_1, \xi_0, \eta_1) + \frac{1}{|\eta_1|} \tilde{a}_1(x_1, \xi_0, \eta_1) + \ldots + \frac{1}{|\eta_1|^N} \tilde{a}_N(x_1, \xi_0, \eta_1), \]
and \( \tilde{a}_i(x_1, \xi_0, \eta_1) \) is homogeneous of degree 0 in \( \eta_1 \) for \( i = 0, 1, \ldots, N \). (The strange \( |\eta_1|^{\frac{3}{2}} \) prefactor will be explained later.) In other words, it is possible to generalize the geometric optics construction to cases in which \( \Lambda_\tilde{q} \) cannot be parameterized by \( x \). For any details that are omitted, one can consult [Esk11] or [EKS99]. The first step in understanding how this works is to analyze an integral \( I(x_0, x_1, \eta_1) \) of the form
\[ I(x_0, x_1, \eta_1) = \int_\mathcal{R} b(x_0, x_1, \xi_0, \eta_1) e^{i|\eta_1|(\tilde{S}(x_1, \xi_0, \eta_1) + x_0\xi_0)} d\xi_0, \]
where \( b(x_0, x_1, \xi_0, \eta_1) \) is homogeneous of degree 0 in \( \eta_1 \) and \( b \) vanishes for \( |\xi_0| \) large. Expanding \( b(x_0, x_1, \xi_0, \eta_1) \) in a Taylor series centered at \( x_0 = \bar{x}_0(x_1, \xi_0, \eta_1) \) yields

\[
b(x_0, x_1, \xi_0, \eta_1) = \sum_{j=0}^{N} \frac{1}{j!} \frac{\partial^j b(\bar{x}_0(x_1, \xi_0, \eta_1), x_1, \xi_0, \eta_1)}{\partial x_0^j} (x_0 - \bar{x}_0(x_1, \xi_0, \eta_1))^j + \mathcal{O}((x_0 - \bar{x}_0(x_1, \xi_0, \eta_1))^{N+1}). \tag{3.7}
\]

Furthermore, note that

\[
\frac{1}{i |\eta_1|} \frac{\partial}{\partial \xi_0} e^{i|\eta_1|(S(x_1, \xi_0, \frac{\eta_1}{|\eta_1|}) + x_0 \xi_0)}
= \left( \frac{\partial \tilde{S}}{\partial \xi_0} \left( x_1, \xi_0, \frac{\eta_1}{|\eta_1|} \right) + x_0 \right) e^{i|\eta_1|(S(x_1, \xi_0, \frac{\eta_1}{|\eta_1|}) + x_0 \xi_0)}
= \left( -\bar{x}_0 \left( x_1, \xi_0, \frac{\eta_1}{|\eta_1|} \right) + x_0 \right) e^{i|\eta_1|(S(x_1, \xi_0, \frac{\eta_1}{|\eta_1|}) + x_0 \xi_0)} \tag{3.8}
\]

As a slight aside we now mention that by using equation (3.8) and integrating by parts several times, it is clear that the contribution of the set

\[
\{ \xi_0 : |x_0 - \bar{x}_0(x_1, \xi_0, \eta_1)| > \epsilon \} \tag{3.9}
\]

to the integral \( I \) for any \( \epsilon > 0 \) is \( \mathcal{O}(\frac{1}{|\eta_1|^N}) \) for all \( N \). Using the fact that \( b(x_0, x_1, \xi_0, \eta_1) = b \left( x_0, x_1, \xi_0, \frac{\eta_1}{|\eta_1|} \right) \) and equations (3.7) and (3.8), we can integrate by parts in \( \xi_0 \) to obtain

\[
I(x_0, x_1, \eta_1) = \int_{\mathbb{R}} \sum_{j=0}^{N} \frac{1}{|\eta_1|^j} \bar{c}_j(x_1, \xi_0, \eta_1) e^{i|\eta_1|(S(x_1, \xi_0, \eta_1) + x_0 \xi_0)} d\xi_0 + \mathcal{O} \left( \frac{1}{|\eta_1|^{N+1}} \right), \tag{3.10}
\]

where \( \bar{c}_0(x_1, \xi_0, \eta_1) = b \left( \bar{x}_0 \left( x_1, \xi_0, \frac{\eta_1}{|\eta_1|} \right), x_1, \xi_0, \frac{\eta_1}{|\eta_1|} \right) \) and \( \bar{c}_j \) for \( j = 1, \ldots, N \) is a more complicated expression that is also homogeneous of degree 0 in \( \eta_1 \). Equation (3.10) provides us with the necessary tool to systematically organize \( Pu \) as a sum of different terms which are grouped together by their degree of homogeneity in \( \eta_1 \). For instance, it is easy to see that the term in \( Pu \) with the highest order with respect to \( \eta_1 \) is

\[
\left( \frac{|\eta_1|}{2\pi} \right)^{\frac{3}{2}} |\eta_1|^2 \int_{\mathbb{R}} p_2 \left( \bar{x}_0 \left( x_1, \xi_0, \frac{\eta_1}{|\eta_1|} \right), x_1, \xi_0, \frac{\partial \tilde{S}}{\partial x_1} \left( x_1, \xi_0, \frac{\eta_1}{|\eta_1|} \right) \right) \times \bar{a}_0(x_1, \xi_0, \eta_1) e^{i|\eta_1|(S(x_1, \xi_0, \eta_1) + x_0 \xi_0)} d\xi_0. \tag{3.11}
\]
Examining equation (3.4), we now see that we picked $\tilde{S}$ precisely so that expression (3.11) would vanish. Furthermore, just as was the case in section 2.2, the fact that $\tilde{S}$ satisfies equation (3.4) insures us that the second highest order term in $Pu$ only involves the unknown $\dot{a}_0$ (the third highest order term will only involve $\dot{a}_1$ and $\ddot{a}_0$ and so on). Using equation (3.10) again, we see that the second highest order term in $Pu$ is

$$
i \left( \frac{|\eta_1|}{2\pi} \right)^\frac{3}{2} |\eta_1| \prod \int_{\mathcal{R}} \left( \frac{\partial}{\partial \xi_0} \left( \tilde{a}_0 \frac{\partial^2 p}{\partial x_0} (\tilde{x}_0, x_1, x_0, \tilde{\xi}_1) \right) - \ddot{a}_0 \frac{\partial^2 \tilde{S}}{\partial x_1} \left( \tilde{x}_0, x_1, \xi_0, \tilde{\xi}_1 \right) - \ddot{a}_0 \frac{\partial^2 p}{\partial x_1 \partial \xi_1} (\tilde{x}_0, x_1, \xi_0, \tilde{\xi}_1) \right)$$

$$\times e^{i|\eta_1| \left( \tilde{S} (x_1, \xi_0, \frac{\eta_1}{|\eta_1|}) + x_0 \xi_0 \right)} d\xi_0,$$

where the arguments $(x_1, \xi_0, \frac{\eta_1}{|\eta_1|})$ are being suppressed in all functions of the form $\tilde{h}$ in the above integrand. In the case of this simple Tricomi example, we obtain the equation

$$0 = \ddot{a}_0 \frac{\partial \tilde{S}}{\partial x_1} \left( \tilde{x}_0, x_1, \xi_0, \tilde{\xi}_1 \right) - \ddot{a}_0 \frac{\partial \tilde{S}}{\partial \xi_0} \left( \tilde{x}_0, x_1, \xi_0, \tilde{\xi}_1 \right)$$

$$= - \ddot{a}_0 \frac{\partial \xi_0}{\partial \xi_0} \left( \tilde{x}_0, x_1, \xi_0, \tilde{\xi}_1 \right) - \ddot{a}_0 \frac{\partial \xi_1}{\partial \xi_1} \left( \tilde{x}_0, x_1, \xi_0, \tilde{\xi}_1 \right),$$

(3.13)

for $\ddot{a}_0$. In the more natural $(s, y_1, |\eta_1|)$ coordinates, of course, this equation is simply

$$\dot{\ddot{a}}_0 (s, y_1, \frac{\eta_1}{|\eta_1|}) = 0.$$

(3.14)

The equations for $\ddot{a}_i$, $i = 1, \ldots, N$ are of the same form, but will have an inhomogeneous term.

We will pick $\ddot{a}_0 \left(0, y_1, \frac{\eta_1}{|\eta_1|}\right)$ to be a smooth function that is equal to 1 for $|y_1| \leq 1$ and 0 for $|y_1| \geq \frac{7}{6}$. (It will become clear later why we picked $\frac{7}{6}$.) We will set $\ddot{a}_i \left(0, y_1, \frac{\eta_1}{|\eta_1|}\right) = 0$ for all $y$ and $i = 1, \ldots, N$. Then $\ddot{a}_i$ will be defined for all $\eta_1$ of size one and hence for all $\eta_1$ since $\ddot{a}_i$ is homogeneous of degree 0 in $\eta_1$. Let us now check that $\ddot{a}_i (x_1, \xi_0, |\eta_1|) = 0$ for $|\xi_0|$ large. This will insure that all of the preceeding calculations beginning with equation (3.6) make sense. Since $\ddot{a}_0 (s, y_1, \frac{\eta_1}{|\eta_1|})$ is constant in $s$ by equation (3.14), $\ddot{a}_0 \left(s, y_1, \frac{\eta_1}{|\eta_1|}\right) = 0$ for $|y_1| \geq \frac{7}{6}$. Since $\dot{\ddot{a}}_1 (x_1, \xi_0, \eta_1) = x_1 - \frac{23}{640}$, it is clear that $\ddot{a}_0 (x_1, \xi_0, \eta_1) = 0$ for $|\xi_0|$ large. Next, note that the inhomogeneous term in the equation for $\ddot{a}_1$ vanishes whenever $\ddot{a}_0$ vanishes.
Thus, since the initial data for $\dot{a}_1$ is 0, $\dot{a}_1$ vanishes whenever $\dot{a}_0$ does. Similarly, $\dot{a}_2$ vanishes whenever both $\dot{a}_1$ and $\dot{a}_0$ vanish. Inductively, then, we can deduce that $\ddot{a}_i(x_1, \xi_0, \eta_1) = 0$ for $|\xi_0|$ large and for $i = 0, \ldots, N$.

3.2 Building the Beam

For the null bicharacteristic given in (3.2), $y_1 = 0$ and $\eta_1 = 1$. We are interested in seeing what a gaussian beam solution $u(x_0, x_1, k)$ centered along the ray path $(\mathcal{x}_0(s), \mathcal{x}_1(s))$ of this null bicharacteristic looks like. Furthermore, since this ray path intersects the line $x_0 = c$ twice for each $c < 0$, we expect $u(c, x_1, k)$ to be the sum of two gaussians centered at these two distinct points of intersection. We reiterate that we cannot use the standard gaussian beam construction from section 2.3 because $(\dot{x}_0(0), \dot{x}_1(0))$ vanishes.

Our approach will instead be to use fourier synthesis to build a gaussian beam solution from the asymptotic solutions constructed above. We define $f(y_1, k)$ to be

$$f(y_1, k) = e^{ik(y_1\eta_1 + \frac{D}{2}y_1^2)},$$

for some $D > 0$ and denote by $\mathcal{F}[f]$ the Fourier transform of $f$ in $y_1$:

$$\mathcal{F}[f](\eta_1, k) = \int_{\mathcal{R}} f(y_1, k)e^{-iy_1\eta_1} dy_1$$

$$= \left(\frac{2\pi}{Dk}\right)^{\frac{1}{2}} e^{-\frac{(\eta_1 - k\eta_1)^2}{2Dk}}.$$  

We then define $u(x_0, x_1, k)$ by

$$u(x_0, x_1, k) = \frac{1}{2\pi} \int_{\mathcal{B}} \mathcal{F}[f](\eta_1, k) u(x_0, x_1, \eta_1) d\eta_1,$$  

(3.15)

where

$$\mathcal{B} = \left\{ \eta_1 \in \mathcal{R} : |\eta_1 - k\eta_1| < k^{\frac{1}{2}+\epsilon} \right\},$$

for some fixed $\epsilon > 0$. From here on out, we deal with $k$ that are large enough so that $B^c$ contains a neighborhood of 0. Therefore, $u$ is well defined and we can differentiate underneath
the integral sign to obtain

\[ Pu(x_0, x_1, k) = \frac{1}{2\pi} \int_{B} F[f](\eta_1, k) Pv(x_0, x_1, \eta_1) d\eta_1. \quad (3.16) \]

Note that for any \( j > 0 \),

\[ \int_{B} F[f](\eta_1, k) |\eta_1|^{-j} d\eta_1 = \left( \frac{2\pi}{D}\right)^{\frac{1}{2}} \int_{B} e^{-\frac{(\eta_1 - k\frac{\eta_1}{|\eta_1|})^2}{2D}} |\eta_1|^{-j} d\eta_1 \]

\[ = \left( \frac{2\pi}{D}\right)^{\frac{1}{2}} \int_{|\zeta| < k^*} e^{-\frac{\zeta^2}{2D}} |k\eta_1 + \sqrt{k}\zeta|^{-j} d\zeta \]

\[ \leq \left( \frac{2\pi}{D}\right)^{\frac{1}{2}} \int_{|\zeta| < k^*} e^{-\frac{\zeta^2}{2D}} \left( k\frac{|\eta_1|}{|\eta_1|} - k^{\frac{1}{2} + \epsilon}\right)^{-j} d\zeta \]

\[ = O \left( k^{-j}\right). \quad (3.17) \]

Therefore, combining equations (3.16) and (3.17), we see that \( Pu \) vanishes to high order in \( k \) if \( Pv \) vanishes to high order in \( |\eta_1| \).

Finally, we would like to see what \( u \) looks like at \((x_0(-1), x_1(-1)) = (-1, -\frac{2}{3})\) and at \((x_0(1), x_1(1)) = (-1, \frac{2}{3})\). We start by applying the stationary phase lemma to \( v(-1, -\frac{2}{3}, \eta_1) \) and consider \( |\eta_1| \) to be the large parameter. In the process, we will see why we included the \( |\eta_1|^\frac{1}{2} \) prefactor term in the definition of \( v \). In what follows, \( x_1 \) will be confined to a ball of radius \( \frac{1}{6} \) centered at \(-\frac{2}{3}\). Let us first find the value(s) of \( \xi_0 \) that make the phase function in the integrand of \( v(-1, x_1, \eta_1) \) stationary. Setting the derivative of \( \hat{S}(x_1, \xi_0, \frac{\eta_1}{|\eta_1|}) + (-1)\xi_0 \) with respect to \( \xi_0 \) equal to 0 yields the equation

\[ \hat{x}_0 \left( x_1, \xi_0, \frac{\eta_1}{|\eta_1|}\right) = -1. \]

It is easy to see using our explicit formula for \( \hat{x}_0 \) that the two solutions to this equation are \( \xi_0 = -1 \) and \( \xi_0 = 1 \). Even though there are two values at which the phase becomes stationary, only one of them will end up mattering since \( \hat{a} \left( x_1, 1, \frac{\eta_1}{|\eta_1|}\right) \) vanishes for the following reason. Note that since we are only dealing with large \( k \), for all \( \eta_1 \in B, \eta_1 \) and \( \frac{\eta_1}{|\eta_1|} \) have the same sign. Since \( \frac{\eta_1}{|\eta_1|} = 1 \) in our example, we can assume that \( \eta_1 > 0 \) in what follows. Therefore, for the small region in the \( x_1 \) axis with which we are dealing, \( |\hat{y} \left( x_1, 1, \frac{\eta_1}{|\eta_1|}\right)| \geq \frac{7}{6} \) which implies
that \( \tilde{a}(x_1, 1, \frac{m}{|m|}) = 0 \) by the remarks following equation (3.14). We are now ready to use the stationary phase lemma to obtain

\[
v(-1, x_1, \eta_1) = \left( \frac{|\eta|}{2\pi} \right)^{\frac{1}{2}} \int_{\mathbb{R}} \tilde{a}(x_1, \xi_0, \eta_1) e^{i|\eta|(|\xi(x_1, \xi_0, \eta_1) - \xi_0|)} d\xi_0
\]

\[
= \left| \text{Det} \frac{\partial \tilde{a}}{\partial \xi_0} (x_1, -1, \frac{\eta_1}{|\eta_1|}) \right|^{-\frac{1}{2}} \tilde{a}(x_1, -1, \eta_1) \\
\times e^{i|\eta|(|\tilde{\xi}(x_1, -1, \frac{\eta_1}{|\eta_1|}) - \xi_0|)} - i\frac{\eta}{|\eta|} \text{Sgn} \frac{\partial \tilde{a}}{\partial \xi_0} (x_1, -1, \frac{\eta_1}{|\eta_1|}) + O(|\eta|^{-1})
\]

\[
= \frac{1}{\sqrt{2}} \tilde{a}(x_1, -1, \eta_1) e^{i|\eta|(|\tilde{\xi}(x_1, -1, \frac{\eta_1}{|\eta_1|}) - \xi_0|)} + O(|\eta|^{-1})
\]

\[
= \frac{1}{\sqrt{2}} \tilde{a}_0(x_1, -1, \eta_1) e^{i|\eta|(|\tilde{\xi}(x_1, -1, \frac{\eta_1}{|\eta_1|}) - \xi_0|)} + O(|\eta|^{-1}), \tag{3.18}
\]

where \( \text{Sgn} \) and \( \text{Det} \) refer to signature and determinant respectively.

Combining equations (3.17) and (3.18) yields

\[
u(-1, x_1, k) = \frac{1}{2\pi} \int_{\mathbb{R}} \mathcal{F}[f](\eta_1, k) \frac{1}{\sqrt{2}} \tilde{a}_0(x_1, -1, \eta_1) e^{i|\eta|(|\tilde{\xi}(x_1, -1, \frac{\eta_1}{|\eta_1|}) - \xi_0|)} d\eta_1 + O(k^{-1})
\]

\[
= \left( \frac{1}{4\pi Dk} \right)^{\frac{1}{2}} e^{-i\frac{\eta}{|\eta|}} \int_{\mathbb{R}} e^{-\frac{(n_k-\eta_k)^2}{2\pi k}} \tilde{a}_0(x_1, -1, \frac{\eta_1}{|\eta_1|}) e^{i\eta(x_1 + \frac{\eta}{|\eta|})} d\eta_1 + O(k^{-1})
\]

\[
= \left( \frac{1}{4\pi Dk} \right)^{\frac{1}{2}} e^{-i\frac{\eta}{|\eta|}} \tilde{a}_0(x_1, -1, 1) \int_{\mathbb{R}} e^{-\frac{(n_k-\eta_k)^2}{2\pi k}} e^{i\eta(x_1 + \frac{\eta}{|\eta|})} d\eta_1 + O(k^{-1})
\]

\[
= \left( \frac{1}{4\pi Dk} \right)^{\frac{1}{2}} e^{-i\frac{\eta}{|\eta|}} e^{i\eta_k(x_1 + \frac{\eta}{|\eta|})} \tilde{a}_0(x_1, -1, 1) \int_{\mathbb{R}} e^{-\frac{\eta^2}{2\pi k}} e^{i\eta(x_1 + \frac{\eta}{|\eta|})} d\eta_1 + O(k^{-1})
\]

\[
= \frac{\tilde{a}_0(x_1, -1, 1)}{\sqrt{2}} e^{-i\frac{\eta}{|\eta|}} e^{i\eta_k(x_1 + \frac{\eta}{|\eta|}) - \frac{\partial k}{\partial x} (x_1 + \frac{\eta}{|\eta|})^2} + O(k^{-1}). \tag{3.19}
\]

Similarly, if \( x_1 \) is in a ball of radius \( \frac{1}{6} \) centered at \( \frac{2}{3} \), we obtain

\[
u(-1, x_1, k) = \frac{\tilde{a}_0(x_1, -1, 1)}{\sqrt{2}} e^{i\frac{\eta}{|\eta|}} e^{i\eta_k(x_1 - \frac{\eta}{|\eta|}) - \frac{\partial k}{\partial x} (x_1 - \frac{\eta}{|\eta|})^2} + O(k^{-1}). \tag{3.20}
\]

Because \( Pu \) vanishes to high order in \( k \) and because equations (3.19) and (3.20) hold, \( u \) is indeed a gaussian beam solution that gets multiplied by the phase factor \( i \) after it passes through the cusp at \((0, 0)\).
To deal with other operators that admit null bicharacteristics with cusps, we will first need to develop some tools. In the preceding example, for instance, for all $\eta_1 \in \mathcal{B}$, $\frac{\eta_1}{|\eta_1|} = 1$ for all $k$. In general, as $k$ increases, $\frac{\tilde{\eta}_1}{|\tilde{\eta}_1|}$ will approach $\frac{\tilde{\eta}}{|\tilde{\eta}|}$. More importantly, $\tilde{y}$ will in general be a more complicated expression and so it will not be possible to explicitly calculate the above inverse fourier transform. In the next chapter, we will develop the necessary tools to tackle these issues and will in the process arrive at new formula for the phase of a gaussian beam which highlights the connection to geometric optics.
CHAPTER 4

The Phase Formula

We start by developing a tool to analyze integrals similar to the one encountered in equation (3.19). In the process, we will arrive at a formula for the phase of a gaussian beam that involves the solution to the eikonal equation of geometric optics. We will then prove that this phase function indeed satisfies the necessary equations.

4.1 Applying a Parametrix to Gaussian Beam Initial Data

In this section, we will continue to use all of the notation that was developed in sections 2.2 and 2.3. Definitions and notation relating to the operator $P$, the solution $\phi$ to the eikonal equation (2.2), the manifold $\Lambda$, and gaussian beams have been introduced in these sections and can be consulted as is necessary. Given a fixed $\tilde{\eta}$ and a positive definite $n \times n$ matrix $D$, we define $f(\tilde{y}, k)$ to be

$$f(\tilde{y}, k) = e^{ik(y_j\eta_j + \frac{1}{2}y_jD_{jl}y_l)},$$

and denote by $\mathcal{F}[f]$ the Fourier transform of $f$ in $\tilde{y}$:

$$\mathcal{F}[f](\tilde{\eta}, k) = \int_{\mathbb{R}^n} f(\tilde{y}, k)e^{-iy_j\eta_j}d\tilde{y}$$

$$= (\text{Det} D)^{-\frac{1}{2}} \left( \frac{2\pi}{k} \right)^\frac{n}{2} e^{-\frac{(\eta_j-k_2_j)}{2k}D_{jl}^{-1}(\eta_j-k_2j)}.$$

The function $f$ is the initial profile of a gaussian beam centered around the null bicharacteristic $(\tilde{x}(s), \tilde{\xi}(s))$ determined by $\tilde{y} = 0$ and $\tilde{\eta} = \tilde{\eta}$. In the spirit of section 3.2, we will build a
gaussian beam \( v \) with initial profile \( f \) by using a parametrix. We define \( v(x, k) \) to be

\[
v(x, k) = \left( \frac{1}{2\pi} \right)^{n/2} \int_{\mathcal{B}} \mathcal{F}[f](\tilde{\eta}, k) e^{i\phi(x, \tilde{\eta})} \, d\tilde{\eta},
\]

where

\[
\mathcal{B} = \left\{ \tilde{\eta} \in \mathbb{R}^n : \|\tilde{\eta} - k\tilde{\eta}\| < k^{\frac{1}{2} + \epsilon} \right\},
\]

for some fixed \( \epsilon > 0 \). We assume that \( k \) is large enough to insure that \( \mathcal{B} \) does not contain the origin. Making the change of variables \( \tilde{\zeta} = \frac{\tilde{\eta} - k\tilde{\eta}}{\sqrt{k}} \), we obtain

\[
v(x, k) = (\text{Det}D)^{-\frac{1}{2}} \left( \frac{1}{2\pi} \right)^{n/2} \int_{|\tilde{\zeta}| < k^\epsilon} e^{-\frac{\zeta_j D_{\tilde{\eta}}^{-1} \zeta_l}{2}} e^{i\phi(x, \sqrt{k}\tilde{\zeta} + k\tilde{\eta})} \, d\tilde{\zeta}.
\]

We know from corollary 2.5 that \( \phi(x, \tilde{\eta}) \) is homogeneous of degree 1 in \( \tilde{\eta} \) and so

\[
\phi \left( x, \sqrt{k}\tilde{\zeta} + k\tilde{\eta} \right) = k \phi \left( x, \tilde{\eta} + \frac{\tilde{\zeta}}{\sqrt{k}} \right).
\]

Then, by Taylor’s theorem,

\[
\phi \left( x, \tilde{\eta} + \frac{\tilde{\zeta}}{\sqrt{k}} \right)
= \phi(x, \tilde{\eta}) + \frac{1}{\sqrt{k}} \frac{\partial \phi}{\partial \eta_j} (x, \tilde{\eta}) \zeta_j + \frac{1}{2k} \zeta_j \zeta_l \frac{\partial^2 \phi}{\partial \eta_j \partial \eta_l} (x, \tilde{\eta}) \zeta_l + \frac{1}{6k^{3/2}} \frac{\partial^3 \phi}{\partial \eta_j \partial \eta_l \partial \eta_m} \left( x, \tilde{\eta} + r \frac{\tilde{\zeta}}{\sqrt{k}} \right) \zeta_j \zeta_l \zeta_m,
\]

where \( |r(x, \tilde{\zeta}, k)| \leq 1 \). Define \( C \) to be the maximum of the magnitude of the third order partials of \( \phi \) on the set

\[
\left\{ (x, \tilde{\eta} + \tilde{\nu}) : |\tilde{\nu}| \leq |\tilde{\eta}| \right\}.
\]
Then, we have that
\[
\left| \int_{|\tilde{\zeta}|<k^*} e^{-\frac{c_j D^{-1}_{jl} \zeta_l}{2}} e^{i\left(k\phi(x,\tilde{\eta})+\sqrt{k} \frac{\partial \phi}{\partial \eta_j}(x,\tilde{\eta})\zeta_j + \frac{1}{2} \zeta_j \frac{\partial^2 \phi}{\partial \eta_i \partial \eta_m}(x,\tilde{\eta})\zeta_i \right)} \left( e^{i \frac{c_{lm} \kappa}{\sqrt{k} \eta_j \partial \eta_l \partial \eta_m}} \frac{\partial^3 \phi}{\partial \eta_j \partial \eta_l \partial \eta_m} \left(x,\tilde{\eta}+r \tilde{\zeta}/\sqrt{k}\right) \zeta_j \zeta_i \zeta_m - 1 \right) d\tilde{\zeta} \right| 
\]
\[
\leq \int_{|\tilde{\zeta}|<k^*} e^{-\frac{c_j D^{-1}_{jl} \zeta_l}{2}} e^{i\left(k\phi(x,\tilde{\eta})+\sqrt{k} \frac{\partial \phi}{\partial \eta_j}(x,\tilde{\eta})\zeta_j + \frac{1}{2} \zeta_j \frac{\partial^2 \phi}{\partial \eta_i \partial \eta_m}(x,\tilde{\eta})\zeta_i \right)} \left( e^{i \frac{c_{lm} \kappa}{\sqrt{k} \eta_j \partial \eta_l \partial \eta_m}} \frac{\partial^3 \phi}{\partial \eta_j \partial \eta_l \partial \eta_m} \left(x,\tilde{\eta}+r \tilde{\zeta}/\sqrt{k}\right) \zeta_j \zeta_i \zeta_m - 1 \right) d\tilde{\zeta} 
\]
\[
\leq \frac{c_{jl}}{\sqrt{k}} \int_{|\tilde{\zeta}|<k^*} e^{-\frac{c_j D^{-1}_{jl} \zeta_l}{2}} \left| \zeta_l \right|^3 d\tilde{\zeta} 
\]
\[
\leq \frac{c_{jl}}{\sqrt{k}} \int_{R^n} e^{-\frac{c_j D^{-1}_{jl} \zeta_l}{2}} \left| \zeta_l \right|^3 d\tilde{\zeta} = \mathcal{O} \left( \frac{1}{\sqrt{k}} \right). 
\]
Using relations (4.2) and (4.3), we obtain
\[
v(x, k) = (\text{Det} D)^{-\frac{1}{2}} \left( \frac{1}{2\pi} \right)^\frac{n}{2} \int_{|\zeta|<k^*} e^{-\frac{c_j D^{-1}_{jl} \zeta_l}{2}} e^{i\left(k\phi(x,\tilde{\eta})+\sqrt{k} \frac{\partial \phi}{\partial \eta_j}(x,\tilde{\eta})\zeta_j + \frac{1}{2} \zeta_j \frac{\partial^2 \phi}{\partial \eta_i \partial \eta_m}(x,\tilde{\eta})\zeta_i \right)} d\zeta 
\]
\[
+ \mathcal{O} \left( \frac{1}{\sqrt{k}} \right)
\]
\[
= \frac{e^{ik\phi(x,\tilde{\eta})}}{\sqrt{\text{Det} D}} \left( \frac{1}{2\pi} \right)^\frac{n}{2} \int_{|\zeta|<k^*} e^{-\frac{c_j (D^{-1}_{jl} - i \frac{\partial^2 \phi}{2 \eta_j \partial \eta_m}) \zeta_l}{2}} e^{i\sqrt{k} \frac{\partial \phi}{\partial \eta_j}(x,\tilde{\eta})\zeta_j} d\zeta + \mathcal{O} \left( \frac{1}{\sqrt{k}} \right)
\]
\[
= \frac{e^{ik\phi(x,\tilde{\eta})}}{\sqrt{\text{Det} D}} \left( \frac{1}{2\pi} \right)^\frac{n}{2} \int_{R^n} e^{-\frac{c_j (D^{-1}_{jl} - i \frac{\partial^2 \phi}{2 \eta_j \partial \eta_m}) \zeta_l}{2}} e^{i\sqrt{k} \frac{\partial \phi}{\partial \eta_j}(x,\tilde{\eta})\zeta_j} d\zeta + \mathcal{O} \left( \frac{1}{\sqrt{k}} \right)
\]
\[
= \frac{1}{\sqrt{\text{Det} D \cdot \text{Det} \left( D^{-1} - i \frac{\partial^2 \phi}{\partial \eta_j \partial \eta_m} \right)}} e^{-k \frac{\partial \phi}{\partial \eta_j}(x,\tilde{\eta}) \left( D^{-1} - i \frac{\partial^2 \phi}{\partial \eta_i \partial \eta_m} \right)_{jl} \frac{\partial \phi}{\partial \eta_l}(x,\tilde{\eta})} + \mathcal{O} \left( \frac{1}{\sqrt{k}} \right)
\]
\[
= \frac{1}{\sqrt{\text{Det} \left( I - iD \frac{\partial^2 \phi}{\partial \eta_j \partial \eta_m} \right)}} e^{ik \left( \phi(x,\tilde{\eta})+\frac{1}{2} \frac{\partial \phi}{\partial \eta_j}(x,\tilde{\eta}) \left( D^{-1} - i \frac{\partial^2 \phi}{\partial \eta_i \partial \eta_m} \right)_{jl} \frac{\partial \phi}{\partial \eta_l}(x,\tilde{\eta}) \right)} + \mathcal{O} \left( \frac{1}{\sqrt{k}} \right),
\]
(4.4)

where $\sqrt{\text{Det}(\cdot)}$ (when considered as a function whose domain is the set of imaginary $n \times n$ matrices with positive definite real part) is the unique continuous branch that sends positive definite matrices to positive numbers. We have evidently arrived at formulas for the leading order amplitude and phase terms for a gaussian beam with initial profile $f$. In the next
section, we will check directly that our phase has the necessary first and second order partials using the results from section 2.3.

4.2 Verifying the Phase Formula

We start by defining $\psi$ as

$$
\psi(x) = \phi(x, \tilde{\eta}) + \frac{i}{2} \frac{\partial \phi}{\partial \eta_j} (x, \tilde{\eta}) \left( D^{-1} - i \frac{\partial^2 \phi}{\partial \eta\partial \eta} (x, \tilde{\eta}) \right)^{-1} \frac{\partial \phi}{\partial \eta_l} (x, \tilde{\eta}).
$$

(4.5)

We recall that

$$
\bar{x}(s) = \hat{x}(s, \tilde{\eta}),
$$

From proposition 2.3, we know that $\frac{\partial \phi}{\partial \eta_j} (x, \tilde{\eta}) = y_j (x, \tilde{\eta})$ and so, in particular,

$$
\frac{\partial \phi}{\partial \eta_j} (\bar{x}(s), \tilde{\eta}) = y_j (\bar{x}(s), \tilde{\eta}) = 0.
$$

This greatly simplifies computing the partials of $\psi$ along $\bar{x}(s)$. For instance, when computing the second order partials, we can completely ignore the term. We first have that

$$
\frac{\partial \psi}{\partial x_j} (\bar{x}(s)) = \frac{\partial \phi}{\partial x_j} (\bar{x}(s), \tilde{\eta}) = \xi_j (s) = \hat{\xi}_j (s, 0, \tilde{\eta})
$$

(4.6)

by proposition 2.3 and so the first order partials of $\psi$ are as they should be according to the gaussian beam construction in section 2.3. If we define

$$
M_{mp}(s) := \frac{\partial^2 \psi}{\partial x_m \partial x_p} (\bar{x}(s)),
$$

we see that

$$
M_{mp}(s) = \frac{\partial^2 \phi}{\partial x_m \partial x_p} (\bar{x}(s), \tilde{\eta}) + i \frac{\partial^2 \phi}{\partial x_m \partial \eta_j} (\bar{x}(s), \tilde{\eta}) \left( D^{-1} - i \frac{\partial^2 \phi}{\partial \eta\partial \eta} (\bar{x}(s), \tilde{\eta}) \right)^{-1} \frac{\partial \phi}{\partial \eta_l} (\bar{x}(s), \tilde{\eta})
$$

$$
= \frac{\partial \xi_m}{\partial \eta_p} (\bar{x}(s), \tilde{\eta}) + i \frac{\partial \xi_m}{\partial \eta_j} (\bar{x}(s), \tilde{\eta}) \left( D^{-1} - i \frac{\partial y}{\partial \eta} (\bar{x}(s), \tilde{\eta}) \right)^{-1} \frac{\partial y_l}{\partial x_p} (\bar{x}(s), \tilde{\eta}).
$$

(4.7)
We will now verify directly that $M$ satisfies equation (2.16) for $i = 0, 1, \ldots, n$. We first take note that

\[
0 = \frac{\partial \hat{y}_i}{\partial \eta_k} \left( s, \tilde{y}, \tilde{\eta} \right) = \frac{\partial y_i}{\partial x_p} \left( x(s), \tilde{\eta} \right) \frac{\partial \hat{x}_p}{\partial \eta_k} \left( s, \tilde{y}, \tilde{\eta} \right) + \frac{\partial y_i}{\partial \eta_k} \left( x(s), \tilde{\eta} \right),
\]

(4.8)

and that

\[
\delta_{li} = \frac{\partial y_i}{\partial y_l} \left( s, \tilde{y}, \tilde{\eta} \right) = \frac{\partial y_i}{\partial x_p} \left( x(s), \tilde{\eta} \right) \frac{\partial \hat{x}_p}{\partial y_l} \left( s, \tilde{y}, \tilde{\eta} \right),
\]

(4.9)

\[
\frac{\partial \hat{\xi}_m}{\partial \eta_k} \left( s, \tilde{y}, \tilde{\eta} \right) = \frac{\partial \xi_m}{\partial x_p} \left( x(s), \tilde{\eta} \right) \frac{\partial \hat{x}_p}{\partial \eta_k} \left( s, \tilde{y}, \tilde{\eta} \right) + \frac{\partial \xi_m}{\partial \eta_k} \left( x(s), \tilde{\eta} \right),
\]

(4.10)

and that

\[
\frac{\partial \hat{\xi}_m}{\partial y_i} \left( s, \tilde{y}, \tilde{\eta} \right) = \frac{\partial \xi_m}{\partial x_p} \left( x(s), \tilde{\eta} \right) \frac{\partial \hat{x}_p}{\partial y_i} \left( s, \tilde{y}, \tilde{\eta} \right).
\]

(4.11)

From equation (4.11), we see that

\[
\frac{\partial \xi_m}{\partial x_p} \left( x(0), \tilde{\eta} \right) = 0.
\]

Similary, using equation (4.9), it is possible to deduce that

\[
\frac{\partial y_i}{\partial x_p} \left( x(0), \tilde{\eta} \right) = \delta_{lp}.
\]

We omit the derivation (in similar fashion) of the equation that shows that

\[
\frac{\partial y}{\partial \eta} \left( x(0), \tilde{\eta} \right) = 0.
\]

It is thus clear that $M_{mp}(0) = iD_{mp}$ for $1 \leq m, p \leq n$. Using equations (4.8), (4.9), (4.10), and (4.11), we see that

\[
M_{mp}(s) \left( \frac{\partial \hat{x}_p}{\partial y_i} \left( s, \tilde{y}, \tilde{\eta} \right) + \frac{\partial \hat{x}_p}{\partial \eta_k} \left( s, \tilde{y}, \tilde{\eta} \right) M_{ki}(0) \right)
\]

\[
= M_{mp}(s) \left( \frac{\partial \hat{x}_p}{\partial y_i} \left( s, \tilde{y}, \tilde{\eta} \right) + i \frac{\partial \hat{x}_p}{\partial \eta_k} \left( s, \tilde{y}, \tilde{\eta} \right) D_{ki} \right)
\]

\[
= \frac{\partial \hat{\xi}_m}{\partial y_i} \left( s, \tilde{y}, \tilde{\eta} \right) + i \left( \frac{\partial \hat{\xi}_m}{\partial \eta_k} \left( s, \tilde{y}, \tilde{\eta} \right) - \frac{\partial \xi_m}{\partial \eta_k} \left( x(s), \tilde{\eta} \right) \right) D_{ki}
\]

\[
+ i \frac{\partial \xi_m}{\partial \eta_j} \left( x(s), \tilde{\eta} \right) \left( D^{-1} - i \frac{\partial y}{\partial \eta} \left( x(s), \tilde{\eta} \right) \right)^{-1} \left( \delta_{li} - i \frac{\partial y_i}{\partial \eta_k} \left( x(s), \tilde{\eta} \right) D_{ki} \right)
\]

\[
= \frac{\partial \hat{\xi}_m}{\partial y_i} \left( s, \tilde{y}, \tilde{\eta} \right) + i \left( \frac{\partial \hat{\xi}_m}{\partial \eta_k} \left( s, \tilde{y}, \tilde{\eta} \right) - \frac{\partial \xi_m}{\partial \eta_k} \left( x(s), \tilde{\eta} \right) \right) D_{ki} + i \frac{\partial \xi_m}{\partial \eta_j} \left( x(s), \tilde{\eta} \right) D_{ji}
\]

\[
= \frac{\partial \hat{\xi}_m}{\partial y_i} \left( s, \tilde{y}, \tilde{\eta} \right) + i \frac{\partial \hat{\xi}_m}{\partial \eta_k} \left( s, \tilde{y}, \tilde{\eta} \right) D_{ki}
\]

(4.12)
and so we have showed (using the notation adopted in section 2.3) that $M(s)\delta x_j(s) = \delta \xi_j(s)$ for $j = 1, \ldots, n$. One shows that $M(s)\dot{x}(s) = \dot{\xi}(s)$ in a similar fashion and we therefore omit this simpler step. We have thus established that $M(s)$ satisfies the matrix Riccati equation of section 2.3 and that $\psi$ has the first and second order partials along $x(s)$ prescribed by the gaussian beam construction. As such, there is an explicit connection between the solution to the eikonal equation of geometric optics and the phase of a gaussian beam.

With the benefit of hindsight, we now present another way of interpreting the formula (4.5) for $\psi(x)$. We start by defining $\theta(x) = \phi(x, \tilde{\eta} + \frac{1}{2} \tilde{\zeta}(x))$, where $\tilde{\zeta}(x)$ is defined implicitly by

$$i D \tilde{y}(x, \tilde{\eta} + \tilde{\zeta}(x)) = \tilde{\zeta}(x). \quad (4.13)$$

To restate what was mentioned in section 2.3, $M(0)$, and hence $iD$, determines an $n + 1$ dimensional subspace of $C^{2n+2}$. This subspace is also the tangent space of the $n + 1$ dimensional manifold

$$\left\{ \left( x, \xi \left( x, \tilde{\eta} + \frac{1}{2} \tilde{\zeta}(x) \right) \right) : x \in \mathcal{R}^{n+1} \right\}$$

at $(x(0), \xi(0))$. Since $\phi(x, \eta)$ is associated with the manifold

$$\left\{ (x, \xi(x, \tilde{\eta})) : x \in \mathcal{R}^{n+1} \right\},$$

it is not unreasonable to hypothesize that $\psi(x)$ and $\theta(x)$ are related.

Indeed, they are. If we modify definition (4.13) slightly to

$$i D \tilde{y}(x, \tilde{\eta}) + i D \frac{\partial \tilde{y}}{\partial \tilde{\eta}}(x, \tilde{\eta}) \tilde{\zeta}(x) = \tilde{\zeta}(x), \quad (4.14)$$

we see that

$$\tilde{\zeta}(x) = i \left( D^{-1} - i \frac{\partial \tilde{y}}{\partial \tilde{\eta}}(x, \tilde{\eta}) \right)^{-1} \tilde{y}(x, \tilde{\eta})$$

$$= i \left( D^{-1} - i \frac{\partial^2 \phi}{\partial \tilde{\eta} \partial \tilde{\eta}}(x, \tilde{\eta}) \right)^{-1} \frac{\partial \phi}{\partial \tilde{\eta}}(x, \tilde{\eta}). \quad (4.15)$$

It is from equation (4.15) clear that $\psi$ is an expansion of $\theta$ in the sense that

$$\psi(x) = \phi(x, \tilde{\eta}) + \frac{1}{2} \frac{\partial \phi}{\partial \tilde{\eta}}(x, \tilde{\eta}) \tilde{\zeta}(x). \quad (4.16)$$
It may be possible to use this method to find formulas for the phase of a Gaussian beam in terms of $\phi$ that have the correct higher order partials (order greater than two) along $x(s)$. 
CHAPTER 5

The Phase Shift

5.1 Generalizing the Tricomi Example

We will use the tools developed in the previous sections to determine the phase shift that occurs when a gaussian beam passes through a cusp. We will briefly review our notation and in the process state our assumptions about the linear partial differential operator $P$ of degree $m$ whose principal symbol is $p_m$. We will assume that there is a smooth function $\eta_0(\tilde{y}, |\tilde{\eta}|)$ such that $p_m\left(0, \tilde{y}, \eta_0(\tilde{y}, |\tilde{\eta}|)\right) = 0$ for all $\tilde{y}$ and $\tilde{\eta}$ in $\mathbb{R}^n$. We recall that in section 2.2 we used the homogeneity of $p_m$ in $\xi$ to show that $\eta_0(\tilde{y}, \tilde{\eta}) := |\tilde{\eta}| \eta_0(\tilde{y}, |\tilde{\eta}|)$ satisfies $p_m(0, \tilde{y}, \eta_0(\tilde{y}, \tilde{\eta}), \tilde{\eta}) = 0$. We denote by $(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta}))$ the null bicharacteristic whose initial $(s = 0)$ value is

$$(\hat{x}(0, \tilde{y}, \tilde{\eta}), \hat{\xi}(0, \tilde{y}, \tilde{\eta})) = (0, \tilde{y}, \eta_0(\tilde{y}, \tilde{\eta}), \tilde{\eta})$$

and fixing $\tilde{y} = 0$ and $\tilde{\eta}$, we recall that

$$(x(s), \xi(s)) = (\hat{x}(s, 0, \tilde{\eta}), \hat{\xi}(s, 0, \tilde{\eta})) = (\hat{x}(s, 0, \tilde{\eta}), \hat{\xi}(s, 0, \tilde{\eta})).$$

We will assume that $\hat{\xi}(0) = 0$ so that there is a cusp in the ray path of this null bicharacteristic at $s = 0$. Recall that $\Lambda_{\tilde{\eta}}$ is the $n+1$ dimensional lagrangian manifold consisting of the null bicharacteristics $(\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta}))$ for fixed $\tilde{\eta} \neq 0$ and that $\Lambda = \bigcup_{\tilde{\eta}} \Lambda_{\tilde{\eta}}$. The notation $x = (x_{(0)}, x_{(1)})$ indicates that the set of coordinates $\{x_0, x_1, \ldots, x_n\}$ is being partitioned into two subsets $x_{(0)}$ and $x_{(1)}$. We will denote the sizes of $x_{(0)}$ and $x_{(1)}$ by $n_0$ and $n_1$ respectively. Then, in context, $\xi_{(0)}$ and $\xi_{(1)}$ are taken to be the analogous partitioning subsets of
\{\xi_0, \xi_1, \ldots, \xi_n\}. Since \( \Lambda_\hat{\xi} \) is a lagrangian manifold, it is a standard fact in symplectic geometry that we can find canonical coordinates \((x(1), \xi(0))\) for \( \Lambda_\hat{\xi} \) near \((x(0), \xi(0))\). Without loss of generality, we can also assume that \((x(1), \xi(0))\) are coordinates for \( \Lambda_\hat{\xi} \) near \((x(s), \xi(s))\) for \(-1 \leq s \leq 1\). We will further assume that \( \Lambda_\hat{\xi} \) can be parameterized by \( x \) near \((x(-1), \xi(-1))\) and \((x(1), \xi(1))\). Therefore, in what follows \( \Lambda \) can be parameterized by \((x(s), \xi(s))\) for \(-1 \leq s \leq 1\) and by \( x \) near \((x(-1), \xi(-1))\) and \((x(1), \xi(1))\). We will thus use the three coordinate systems \((s, \tilde{y}, \tilde{\eta})\), \((x, \eta)\), and \((x(1), \xi(0), \tilde{\eta})\) for \( \Lambda \) at various points in the following discussion. For a function \( \hat{h} \) defined on \( \Lambda \), we will adopt the convention that

\[
\hat{h}(s, \tilde{y}, \tilde{\eta}) = h(x, \tilde{\eta}) = \hat{h}(x(1), \xi(0), \tilde{\eta}),
\]

whenever the respective coordinate systems makes sense. Indeed, any equations in what follows involving the various coordinate systems will only make sense when these coordinates are valid so we will no longer include this qualifier. Of course, \((s, \tilde{y}, \tilde{\eta})\) are global coordinates on \( \Lambda \) and so any function \( \hat{h}(s, \tilde{y}, \tilde{\eta}) \) is globally defined.

We now proceed as we did in chapter 3. We define \( \hat{S}(s, \tilde{y}, \tilde{\eta}) = \tilde{y} \cdot \tilde{\eta} - \hat{x}(0)(s, \tilde{y}, \tilde{\eta}) \cdot \hat{\xi}(0)(s, \tilde{y}, \tilde{\eta}) \) and note that \( \hat{S}(x(1), \xi(0), \tilde{\eta}) \) satisfies

\[
p_m \left( -\frac{\partial \hat{S}}{\partial \xi(0)} (x(1), \xi(0), \tilde{\eta}) ; x(1), \xi(0), \frac{\partial \hat{S}}{\partial x(1)} (x(1), \xi(0), \tilde{\eta}) \right) = 0.
\] (5.1)

A formal justification for equation (5.1) is given by

\[
d\hat{S} = d \left( \tilde{y} \cdot \tilde{\eta} \right) - d \left( \hat{x}(0) \cdot \xi(0) \right)
= \tilde{y} d\tilde{\eta} + \tilde{\eta} d\tilde{y} - \hat{x}(0) d\xi(0) - \xi(0) d\hat{x}(0)
= \tilde{y} d\tilde{\eta} + \hat{\xi} d\hat{x} - \hat{x}(0) d\xi(0) - \xi(0) d\hat{x}(0)
= \tilde{y} d\tilde{\eta} + \hat{\xi}(1) d\xi(1) - \hat{x}(0) d\xi(0),
\]

which implies that

\[
\frac{\partial \hat{S}}{\partial \tilde{\eta}} (x(1), \xi(0), \tilde{\eta}) = \tilde{y} (x(1), \xi(0), \tilde{\eta}),
\]

\[
\frac{\partial \hat{S}}{\partial x(1)} (x(1), \xi(0), \tilde{\eta}) = \hat{\xi}(1) (x(1), \xi(0), \tilde{\eta}), \quad \text{and}
\]

\[
\frac{\partial \hat{S}}{\partial \xi(0)} (x(1), \xi(0), \tilde{\eta}) = -\hat{x}(0) (x(1), \xi(0), \tilde{\eta})
\] (5.2)
as needed. This argument can be made rigorous with arguments very similar to those found in proposition 2.3.

Pick \( \hat{a}_0 \left( 0, \tilde{y}, \frac{\eta}{|\eta|} \right) \) to be a smooth function whose compact support contains \( \left( \tilde{y}, \frac{\eta}{|\eta|} \right) \) and then define
\[
\hat{a}_0 \left( s, \tilde{y}, \frac{\eta}{|\eta|} \right) = \hat{a}_0 \left( 0, \tilde{y}, \frac{\eta}{|\eta|} \right)
\]
as indicated by equation (3.14). Then \( \hat{a}_0 \) is defined for all unit \( \tilde{\eta} \) and we extend this definition to all \( \tilde{\eta} \) by declaring \( \hat{a}_0 \) to be homogeneous of degree 0 in \( \tilde{\eta} \). Given a positive definite \( n \times n \) matrix \( D \), we define \( f(\tilde{y}, k) \) to be
\[
f(\tilde{y}, k) = e^{ik(y_j \eta_j + \frac{1}{2} y_j D_{jl} y_l)},
\]
and denote by \( \mathcal{F}[f] \) the Fourier transform of \( f \) in \( \tilde{y} \):
\[
\mathcal{F}[f](\tilde{\eta}, k) = \int_{\mathbb{R}^n} f(\tilde{y}, k)e^{-iy_j \eta_j} d\tilde{y}
= (\text{Det} D)^{-\frac{1}{2}} \left( \frac{2\pi}{k} \right)^n e^{-\frac{1}{2} \eta_j (\eta_j - k\eta_j) D_{jl}^{-1} (\eta_l - k\eta_l)}.
\]

We now define \( u(x, k) \) by
\[
u(x, k) = \left( \frac{1}{2\pi} \right)^n \int_{\mathcal{B}} \mathcal{F}[f](\tilde{\eta}, k) \left( \frac{|\tilde{\eta}|}{2\pi} \right)^\frac{m_0}{2} \int_{\mathcal{R}^n} \tilde{a}_0(x(1), \xi(0), \tilde{\eta}) e^{i\tilde{\eta}(S(x(1), \xi(0), \frac{\eta}{|\eta|}) + x(0) \cdot \xi(0))} d\xi(0) d\tilde{\eta},
\]
where
\[
\mathcal{B} = \left\{ \tilde{\eta} \in \mathcal{R}^n : |\tilde{\eta} - k\tilde{\eta}| < k^{\frac{1}{2} + \epsilon} \right\},
\]
for some fixed \( \epsilon > 0 \). From here on out, we deal with \( k \) that are large enough so that \( \mathcal{B}^c \) contains a neighborhood of 0. We know from the arguments in section 3.2 that by inserting more amplitude terms \( \tilde{a}_1, \ldots, \tilde{a}_N \) into the above formula for \( u \), we can create an asymptotic solution to \( Pu = 0 \). Since we are only interested in the leading order term in

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Therefore, by equation (3.17), we obtain that

$k$ of this asymptotic solution, and since none of these higher amplitude terms contribute to this leading order term, we omit them from the definition of $u$.

The phase in the inner integrand is stationary when $\tilde{x}_{(0)} \left( x_{(1)}, \xi_{(0)}, \frac{\tilde{\eta}}{|\tilde{\eta}|} \right) = x_{(0)}$. By picking the initial data for $\tilde{a}_0$ properly, we can insure that near $\left( x(-1), \frac{\tilde{\eta}}{|\tilde{\eta}|} \right)$, there is only one solution $\xi_{(0)}^\sim \left( x, \frac{\tilde{\eta}}{|\tilde{\eta}|} \right)$ to $\tilde{x}_{(0)} \left( x_{(1)}, \xi_{(0)}^\sim, \frac{\tilde{\eta}}{|\tilde{\eta}|} \right) = x_{(0)}$ such that $\tilde{a}_0 \left( x_{(1)}, \xi_{(0)}^\sim \left( x, \frac{\tilde{\eta}}{|\tilde{\eta}|} \right), \tilde{\eta} \right) \neq 0$. We define $\xi_{(0)}^+ \left( x, \frac{\tilde{\eta}}{|\tilde{\eta}|} \right)$ analogously. Applying the stationary phase lemma and using equation (3.17), we thus obtain that for $x$ near $x(-1)$

\[
\begin{align*}
    u(x, k) &= \int_{B} \mathcal{F}[f](\tilde{\eta}, k) \frac{\tilde{a}_0 \left( x_{(1)}, \xi_{(0)}^\sim, \frac{\tilde{\eta}}{|\tilde{\eta}|} \right)}{\sqrt{\text{Det} \frac{\partial \tilde{x}_{(0)}}{\partial \xi_{(0)}} \left( x_{(1)}, \xi_{(0)}^\sim, \frac{\tilde{\eta}}{|\tilde{\eta}|} \right)}} e^{i\tilde{\eta} \left( \tilde{\eta} \left( x_{(1)}, \xi_{(0)}^\sim, \frac{\tilde{\eta}}{|\tilde{\eta}|} \right) \right)} - i\tilde{\eta} \bigg| \nabla_{\xi_{(0)}} \bigg( x_{(1)}, \xi_{(0)}^\sim, \frac{\tilde{\eta}}{|\tilde{\eta}|} \bigg) \bigg| d\tilde{\eta} \\
    &\quad + \mathcal{O} \left( \frac{1}{k} \right),
\end{align*}
\]

where $\xi_{(0)}^\sim$ is evaluated at $\left( x, \frac{\tilde{\eta}}{|\tilde{\eta}|} \right)$ in the above expression. For large enough $k$,

\[
\text{Sgn} \frac{\partial \tilde{x}_{(0)}}{\partial \xi_{(0)}} \left( x_{(1)}, \xi_{(0)}^\sim, \frac{\tilde{\eta}}{|\tilde{\eta}|} \right) = \text{Sgn} \frac{\partial \tilde{x}_{(0)}}{\partial \xi_{(0)}} \left( x_{(1)}, \xi_{(0)}^\sim \left( x, \frac{\tilde{\eta}}{|\tilde{\eta}|} \right), \frac{\tilde{\eta}}{|\tilde{\eta}|} \right) =: \Delta^-(x)
\]

for all $\tilde{\eta} \in B$. Furthermore, noting that

\[
\frac{\tilde{a}_0 \left( x_{(1)}, \xi_{(0)}^\sim, \tilde{\eta} \right)}{\sqrt{\text{Det} \frac{\partial \tilde{x}_{(0)}}{\partial \xi_{(0)}} \left( x_{(1)}, \xi_{(0)}^\sim, \tilde{\eta} \right)}}
\]

is homogeneous of degree 0 in $\tilde{\eta}$, we can Taylor expand to obtain

\[
\frac{\tilde{a}_0 \left( x_{(1)}, \xi_{(0)}^\sim, \tilde{\eta} \right)}{\sqrt{\text{Det} \frac{\partial \tilde{x}_{(0)}}{\partial \xi_{(0)}} \left( x_{(1)}, \xi_{(0)}^\sim, \tilde{\eta} \right)}} = \frac{\tilde{a}_0 \left( x_{(1)}, \xi_{(0)}^\sim \left( x, \frac{\tilde{\eta}}{|\tilde{\eta}|} \right), \tilde{\eta} \right)}{\sqrt{\text{Det} \frac{\partial \tilde{x}_{(0)}}{\partial \xi_{(0)}} \left( x_{(1)}, \xi_{(0)}^\sim \left( x, \frac{\tilde{\eta}}{|\tilde{\eta}|} \right), \frac{\tilde{\eta}}{|\tilde{\eta}|} \right)}} + \mathcal{O} \left( \frac{1}{|\tilde{\eta}|} \right)
\]

\[
=: a^- (x) + \mathcal{O} \left( \frac{1}{|\tilde{\eta}|} \right).
\]

Therefore, by equation (3.17), we obtain that

\[
\begin{align*}
    u(x, k) &= a^- (x) e^{-i\tilde{\eta} \Delta^-(x)} \left( \frac{1}{2\pi} \right)^n \int_{B} \mathcal{F}[f](\tilde{\eta}, k) e^{i\tilde{\eta} \cdot (x, \tilde{\eta})} d\tilde{\eta} + \mathcal{O} \left( \frac{1}{k} \right),
\end{align*}
\]

\[38\]
where
\[
\phi^-(x, \tilde{\eta}) := \left| \tilde{\eta} \right| \left( \hat{y}(x(1), \xi_0, \tilde{\eta}) \cdot \frac{\tilde{\eta}}{|\tilde{\eta}|} \right) \quad (5.8)
\]
is homogeneous of degree one in \( \tilde{\eta} \). Note that \( \phi^-(x, \tilde{\eta}) = \tilde{y}^-(x, \tilde{\eta}) \cdot \tilde{\eta} \), where \( s^-(x, \tilde{\eta}) \) and \( \tilde{y}^-(x, \tilde{\eta}) \) are the solutions of
\[
\hat{x}(s^-, \tilde{y}^-, \tilde{\eta}) = x
\]
near \((x(-1), \tilde{\eta})\). From equation (4.4), we thus have that near \( x(-1) \),
\[
u(x,k) =
\frac{a^-(x)}{\sqrt{\text{Det} \left( \mathbf{I} - iD^{-1} \frac{\partial^2 \phi^-}{\partial \eta \partial \eta}(x, \tilde{\eta}) \right)}} e^{ik \left( \phi^-(x, \tilde{\eta}) + \frac{1}{2} \frac{\partial \phi^-}{\partial \eta}(x, \tilde{\eta}) \right) (D^{-1} - i \frac{\partial^2 \phi^-}{\partial \eta \partial \eta} \frac{\partial}{\partial \eta})(x, \tilde{\eta})} + \mathcal{O} \left( \frac{1}{\sqrt{k}} \right).
\]
(5.9)

It is clear that we obtain an analogous formula for \( \nu(x,k) \) near \( x(1) \) where all the negative signs in formula (5.9) are replaced by positive signs and \( \Delta^+(x) \), \( a^+(x) \), and \( \phi^+(x, \tilde{\eta}) \) are defined in the obvious way. We see that the gaussian beam gets multiplied by the phase factor \( e^{-i \frac{\pi}{4} (\Delta^+(x(1)) - \Delta^-(x(-1)))} \) after it passes through the cusp. After each passage through such a cusp in its ray path, the gaussian beam gets multiplied by another such factor. The product of all these factors can be written as \( e^{i \frac{\pi}{4} L} \), and \( L \) is known as the Maslov index of the null bicharacteristic \((x(s), \xi(s))\).

5.2 Future Directions

We are interested in extending the above results to the semiclassical setting. Although the principal symbol of a semiclassical operator will not be homogeneous in \( \xi \), many of the above ideas will still be helpful in deriving the amplitude shift across a cusp. The heuristic at the end of section 4.2 seems to indicate that the homogeneity of \( p_m \) in \( \xi \) is not necessary. A more natural setting in which cusps occur is when one deals with stable periodic orbits in the bicharacteristic flow. In this case, Gaussian beams can be used to build an approximate sequence of eigenfunctions known as a quasimode [Ral79]. These approximate
eigenfunctions become increasingly concentrated along the sole periodic orbit. Quasimodes have applications in differential geometry, geometric optics, and quantum mechanics. We hope to use the above findings to construct quasimodes for a class of operators that was previously unamenable to gaussian beam techniques.
CHAPTER 6

Glossary Of Notation

Conventions for Variables

\[ x = (x(0), x(1)) \]

\[ \tilde{x} = (x_1, \ldots, x_n) \]

a partition of the coordinates into two subsets, e.g. \( x(0) = \{x_1, x_2, x_5\} \) and \( x(1) = \{x_3, x_4, x_6, \ldots, x_n\} \)

\[ x \]

Manifold Related

\[ p_m(x, \xi) \]

principal symbol of the operator \( P \) and the generator of the Hamiltonian flow

\[ s \]

the curve parameter for a bicharacteristic

\[ \eta_0(\tilde{y}, \tilde{\eta}) \]

a solution to \( p_m(0, \tilde{y}, \eta_0, \tilde{\eta}) = 0 \) that is homogeneous of degree 1 in \( \tilde{\eta} \)

\[ (\hat{x}(s, \tilde{y}, \tilde{\eta}), \hat{\xi}(s, \tilde{y}, \tilde{\eta})) \]

the null bicharacteristic passing through \( (0, \tilde{y}, \eta_0(\tilde{y}, \tilde{\eta}), \tilde{\eta}) \)

\[ (\hat{x}(s), \hat{\xi}(s)) \]

\[ \Lambda_{\tilde{\eta}} \]

the collection of null bicharacteristics

\[ \Lambda \quad \cup_{\tilde{\eta}} \Lambda_{\tilde{\eta}} \]
### Conventions for Functions on $\Lambda$

<table>
<thead>
<tr>
<th>Function $f$</th>
<th>$f$ is a function of $(x, \bar{\eta})$</th>
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</thead>
<tbody>
<tr>
<td>Function $\dot{f}$</td>
<td>$\dot{f}$ is a function of $(s, \bar{y}, \bar{\eta})$</td>
</tr>
<tr>
<td>Function $\ddot{f}$</td>
<td>$\ddot{f}$ is a function of $(x_{(1)}, \xi_{(0)}, \bar{\eta})$</td>
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REFERENCES


