Inverse Spectral Problems in Rectangular Domains

Joint work with Gregory Eskin
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We consider the Schrödinger operator,

$$H = -\Delta + q(x),$$

in the rectangular domain

$$R_0 = \{x \in R^n : \ 0 \leq x_i \leq a_i, \ i = 1, .., n\}$$

and on each of the 2^n faces of ∂R_0, $R_0 \cap \{x_i = 0 \text{ or } x_i = a_i\}$ we impose either Dirichlet or Neumann boundary conditions.

Let

$$\mu_1 \leq \mu_2 \leq \mu_3 \leq \cdots$$

be the spectrum with multiplicity of H with these boundary conditions.

We take R and $\{\mu_j\}_{j=1}^{\infty}$ as given, and study what constraints these data put on $q(x)$.

The nature of these constraints depends very much on the class of $q(x)$ that one considers. We consider the following very restricted class $\mathcal{Q}$:

Let

$$R_1 = \{x \in R^n : \ |x_i| \leq a_i, \ i = 1, .., n\}$$

and extend q to R_1 by requiring $q(\pm x_1, \pm x_2, .., \pm x_n) = q(x)$ for $x \in R$. Then extend q to be periodic on R^n by

$$q(x_1 + 2m_1 a_1, x_2 + 2m_2 a_2, .., x_n + 2m_n a_n) = q(x), x \in R_1.$$

We define $\mathcal{Q}$ to the set of q for which this extension is an analytic function for $x \in R^n$.

Equivalently, $\mathcal{Q}$ is the set of q of the form

$$q(x) = \sum_{m \in N^n} a_m \Pi_{j=1}^n \cos(\frac{\pi m_j x_j}{a_j}),$$

where $|a_m| < C \exp(-\delta |m|)$, $\delta > 0$.

Theorem 1: When the numbers $\{a_1^2, .., a_n^2\}$ are rationally independent, there are no continuous deformations of $q(x)$ in $\mathcal{Q}$ which preserve $\{\mu_1, \mu_2, ...\}$.

In the 1980's Eskin, Trubowitz and I considered this question for real analytic potentials on R^n/L, when L is a vector lattice – periodic potentials. In that setting one sees easily that Theorem 1 does not hold. Likewise, if one enlarges $\mathcal{Q}$ to include all square-integrable potentials, it is well-known that Theorem 1 does not hold for rectangular domains. However, as in that earlier work, the results on the features of $q(x)$ that are determined by $\{\mu_1, \mu_2, ...\}$ carry more information than the "rigidity" result in Theorem 1.

To describe the structure determined by $\{\mu_1, \mu_2, ..\}$ we need an expansion of q as a sum of "directional" potentials. In its extended form q is periodic with respect to the lattice L generated by integer linear combinations of the vectors $v_i = 2a_i\hat{e}_i,\ i = 1, .., n$.

To L we associate the dual lattice

$$L^* = \{\delta \in R^n : \delta \cdot d \in Z \text{ for all } d \in L\},$$

and expand q in a Fourier series

$$q(x) = \sum_{\delta \in L^*} a_\delta e^{2\pi i \delta \cdot x}.$$

Subtracting a constant from q, we assume that $a_{(0,..,0)}$ $= 0$. Let S denote the set of elements of S which are maximal in the sense that $\{\delta \cdot d,\ d \in L\} = Z$ [this is simply the set of elements in L which are closest to the origin on the line joining them to the origin]. Then we have the expansion

$$q(x) = \frac{1}{2} \sum_{\delta \in S} \Big(\sum_{k=-\infty}^{\infty} a_{k\delta} e^{2\pi i k \delta \cdot x} \Big) = \frac{1}{2} \sum_{\delta \in S} q_\delta(\delta \cdot x),$$

where

$$q_\delta(s) = \sum_{k=-\infty}^{\infty} a_{k\delta} e^{2\pi i k s}.$$

Note that $q_{-\delta}(s) = q_\delta(-s)$ which gives rise to the factor 1/2 in the expansion. Since $q(x) = q(-x)$, we also have $q_\delta(s) = q_\delta(-s)$.

Theorem 2: Suppose that $q,\ \tilde{q} \in \mathcal{Q}$ are isospectral potentials on R and that $\{a_1^2, \cdots, a_n^2\}$ are rationally independent. Then for each $\delta \in S$ with more than one nonzero component $-|\delta|^2 d^2/ds^2 + q_\delta(s)$ and $-|\delta|^2 d^2/ds^2 + \tilde{q}_\delta(s)$ have the same periodic spectrum on $[-1/2, 1/2]$.

Since $q_\delta(s)$ and $\tilde{q}_\delta(s)$ are even functions of s, this implies that $\tilde{q}_\delta$ with the periodic spectrum as q_δ is either finite or a Cantor set.

There is an extensive literature on multi-dimensional inverse spectral problems. The spectrum of $-\Delta + q$ has been considered on negatively curved two dimensional manifolds by Guillemin and Kazhdan, two-dimensional tori and spheres by Guillemin and by Gordon and Kappeler, and on flat tori by O. Veliev and by Eskin, Trubowitz and myself. Nonetheless, definitive results at the level of M.G. Krein's work on the weighted string in the 1950's remain out of reach.

The proofs of Theorems 1 and 2 follow these steps:

a) Construct the fundamental solution $E_0(t,x,y)$ for the initial value problem $u(0,x) = f(x),\ u_t(0,x) = 0$,

$$u_{tt} = \Delta u - qu \text{ in } R_0$$

with the given boundary conditions from the fundamental solution for this problem (without boundary conditions) in R^n by the "Method of Reflection". The distribution

$$\sum_{j=1}^{\infty} \cos(t\sqrt{\mu_j})$$

is equal to the distribution

$$\int_{R_0} E_0(t,x,x)dx = 2^{-n}\int_{R_1} E_0(t,x,x)dx.$$

For this reason

$$\sum_{j=1}^{\infty} \cos(t\sqrt{\mu_j})$$

is called the "wave trace" for $-\Delta + q(x)$ on R_0 with the given boundary conditions. So $\{\mu_1, \mu_2, ..\}$ determines $\int_{R_1} E_0(t,x,x)dx$.

b) Show that

$$\int_{R_1} E_0(t,x,x)dx$$

is a sum of distributions

$$\sum_{d\in L} D_d(t),$$

where $D_d(t) = 0$ for $|t| < |d|$ and $D_d(t)$ is real analytic in t for $t > |d|$. The hypothesis that $\{a_1^2, .., a_n^2\}$ is linearly independent over the rationals enters here (with some symmetry considerations) to imply that $\int_{R_0} E_0(t,x,x)dx$ determines the $D_d(t)$, $d \in L$.

c) Pass from wave equation traces to the heat equation traces to complete the proof of Theorem 2. This step closely follows much earlier work (Eskin, Ralston, Trubowitz, CPAM **37**(1984)).

d) Give an additional rigidity argument, depending essentially on analyticity, to show of the q_δ admit no continuous isospectral deformations when δ has only one nonzero component.

To describe these steps in more detail I restrict to the special case $n = 2$ with Dirichlet conditions on all four sides of R_0 which will now be just

$$R = \{(x_1, x_2) : 0 \leq x_1 \leq a,\ 0 \leq x_2 \leq b\}.$$

In this case the fundamental solution E_0 given by

$$\begin{gathered} E_0(t, x, y) = \\ \textstyle\sum\limits_{d \in L} E(t, d + x, y) \\ - \textstyle\sum\limits_{d \in L} [E(t, d_1 - x_1, d_2 + x_2, y) + E(t, d_1 + x_1, d_2 - x_2, y)] \\ + \textstyle\sum\limits_{d \in L} E(t, d - x, y). \end{gathered}$$

Here I use functional notation for distributions. All these formulas hold in the conventional sense when one replaces $E(t, x, y)$ by

$$K_\rho(x, y) = \textstyle\int \rho(t) E(t, x, y) dt,$$

$\rho \in C_c^\infty$, but that makes the notation less illuminating. Note that the sums in the expansion of E_0 are finite for x and y in compact sets by finite propagation speed for the wave equation.

The main argument in step b) amounts to showing that when one passes to traces, the terms in the sums defining E_0 become much simpler. The contribution of the first group of terms to $\int_{R_1} E_0(t, x, x)dx$ is

$$\sum_{d \in L} \int_{R_1} E(t, d + x, x).$$

This is the wave trace for $-\Delta + q$ on R_1 with periodic boundary conditions ($u(x+d) = u(x),\ d \in L$). This is exactly what one wants for step c). The essential observation is that, when d has no zero components, the other terms in the expansion are real analytic near $|t| = |d|$. Thus for d without zero components

$$D_d(t) = \sum_{\{\tilde{d}: \tilde{d} = (\pm d_1, \pm d_2)\}} \int_{R_1} E(x + \tilde{d}, x)dx,$$

and by symmetry the four terms in this sum are identical. The contributions of the other terms "telescope" into terms without singularities near $t = \pm|d|$.

Telescoping argument: One can show that

$$\sum_{d\in L}\int_{R_1} E(t,d_1-x_1,d_2+x_2;x)dx$$

$$= \sum_{(0,d_2)\in L}\int_{-b}^{b}\left(\int_{-\infty}^{\infty} E(t,-x_1,d_2+x_2;x)dx_1\right)dx_2 +$$

$$\int_{-b}^{b}\left(\int_{-\infty}^{\infty} E(t,2a-x_1,d_2+x_2;x)dx_1\right)dx_2 \qquad (1)$$

and

$$\sum_{d\in L}\int_{R_1} E(t,d_1+x_1,d_2-x_2;x)dx$$

$$= \sum_{(d_1,0)\in L}\int_{-a}^{a}\left(\int_{-\infty}^{\infty} E(t,d_1+x_1,-x_2;x)dx_2\right)dx_1 +$$

$$\int_{-a}^{a}\left(\int_{-\infty}^{\infty} E(t,d_1+x_1,2b-x_2;x)dx_2\right)dx_1 \qquad (2)$$

and

$$\sum_{d\in L}\int_{R_1} E(t,d-x,x)dx$$

$$=$$

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(t,-x,x)dx+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(t,2a-x_1,x_2;x)dx$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(t,x_1,2b-x_2;x)dx$$

$$+\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} E(t,2a-x_1,2b-x_2;x)dx. \qquad (3)$$

These are the formulas that we need: each term on the right hand side of (1) is zero for $|t| < |d_2|$, and one can show that it is real analytic for $|t| > |d_2|$. Each term on the right hand side of (2) is zero for $|t| < |d_1|$ and real analytic for $|t| > |d_1|$. Each term on the right hand side of (3) is real analytic for $|t| > 0$.

One can relate the terms on the right hand sides of these formulas to the geometry of the rectangle in a useful way. One thinks of the terms in the reduced form of

$$\sum_{d\in L} \int_{R_1} E(t, d_1 - x_1, d_2 + x_2; x)dx$$

as coming from the sides $x_1 = 0$ and $x_1 = a$, those from

$$\sum_{d\in L} \int_{R_1} E(t, d_1 + x_1, d_2 - x_2; x)dx$$

as coming from the sides $x_2 = 0$ and $x_2 = b$, and those from

$$\sum_{d\in L} \int_{R_1} E(t, d - x, x)dx$$

as coming from the corners. This pattern extends to the n-dimensional case.

At the start of step c) one knows that each of the terms

$$\int_{R_1} E(x+d,x)dx$$

where d has no nonzero components is determined by the wave trace, and hence by $\{\mu_1, \mu_2, ..\}$. To reduce this to information on the directional potentials we use the transformation

$$\frac{1}{\sqrt{4\pi t}}\int_{-\infty}^{\infty} e^{-s^2/4t} E(s,d{+}x,x)ds = \int_{R_1} G(t,d{+}x,x)dx,$$

where $G(t,x,y)$ is the fundamental solution for the initial value problem

$$u_t = \Delta u - Qu,\ u(0,x) = f(x), x \in R^2.$$

[We learned this transformation from the work of Y. Kannai.]

From the asymptotic form of $\int_{R_1} G(t,Nd{+}e{+}x,x)dx$ as $N \to \infty$ one recovers $\int_{R_1} G_d(t,e+x,x)dx$, where $G_d(t,x,y)$ is the fundamental solution for the initial value problem for

$$u_t = \Delta u - q_d u,$$

and $q_d(x) = \int_0^1 q(x+sd)ds$. I will not give the details on that here. In two dimensions $q_d(x) = q_\delta(\delta \cdot x)$, where $\delta \in S$ satisfies $\delta \cdot d = 0$. In the higher dimensional cases one simply iterates this part of the until the potential becomes a q_δ.

Knowing that the spectrum determines $\int_{R_1} G_d(t, e + x, x)dx$, $e \in L$, one easily completes the proof of Theorem 2.

The proof of Theorem 1 requires further analysis of the contributions to the traces from terms where d has just one nonzero component. These determine the spectrum of one-dimensional Schrödinger operators on the intervals $[0, a_i], i = 1, .., n$, with Dirichlet or Neumann conditions at the endpoints according the the original assignment of boundary conditions in the definition of the μ_j's. It was a surprise that when the potentials q_δ come from potentials in $\mathcal{Q}$ these operators have discrete isospectral sets. The final lemma in the proof is that the spectrum of such operators with either Dirichlet or Neumann boundary conditions at the endpoints determines their periodic spectrum on the doubled interval $[-a_i, a_i]$ and this again implies that the isospectral sets are either finite or Cantor sets.

The reduced form of the wave trace is convenient for computing its behavior as t goes to zero, and one gets geometrical interpretations of the terms in the formulas. For example for $R_0 = \{x \in R^3 : 0 \leq x_i \leq a_i,\ i = 1,2,3\}$ with Dirichlet boundary conditions on all faces we have the following. Let $c_i = a_i$ or 0, and use Γ_{c_i} for the face $x_i = c_i$, $\Gamma_{c_i c_j}$ for the edge $x_i = c_i,\ x_j = c_j$ and $P_{c_1 c_2 c_3}$ for the vertex $x_1 = c_1,\ x_2 = c_2,\ x_3 = c_3$. The trace $2^3 \int_{R_0} E_0(t,x,x)dx$ is a sum of the following terms: Modulo contributions which are O(t^3) these are:

$$\gamma_0 \delta''(t)|R_0| + \gamma_1 \frac{d}{dt} p.v.\frac{1}{t} \sum_{c_i} |\Gamma_{c_i}| +$$

$$\delta(t)\left[\gamma_{21} \int_{R_0} q(x)dx + \gamma_{22} \sum_{c_i,c_j} |\Gamma_{c_i c_j}|\right]$$

$$+ \left[\gamma_{31} \sum_{c_i} \int_{\Gamma_{c_i}} q ds + \gamma_{32} \sum_{c_i,c_j,c_k} |P_{c_i c_j c_k}|\right]$$

$$+|t|\left[\gamma_{41} \int_{R_0} q^2(x)dx + \gamma_{42} \sum_{c_i,c_j} \int_{\Gamma_{c_i c_j}} q ds\right]$$

$$+t^2 \left[\gamma_{51} \sum_{\Gamma_{c_i}} \int_{\Gamma_{c_i}} q^2 ds + \gamma_{52} \sum_{\Gamma_{c_i}} \int_{\Gamma_{c_i}} \frac{\partial^2}{\partial n^2} q ds + \gamma_{53} \sum_{c_1, c_2, c_3} q(P_{c_1 c_2 c_3}) \right]$$

With a little more effort one can compute the coefficients in preceding formula. They are $\gamma_0 = -(2\pi)^{-1}$, $\gamma_1 = -(8\pi)^{-1}$, $\gamma_{21} = -(4\pi)^{-1}$, $\gamma_{22} = (16)^{-1}$, $\gamma_{31} = (16\pi)^{-1}$, $\gamma_{32} = -(64)^{-1}$, $\gamma_{41} = (32\pi)^{-1}$, $\gamma_{51} = -(64)^{-1}$, $\gamma_{52} = (128\pi)^{-1}$ and $\gamma_{53} = (128)^{-1}$.

Thank you for your attention.