Comments on Assignment 1

1. Suppose that (χ, d) is a metric space and $\{x_n\}_{n=1}^{\infty}$ is a sequence in χ converging to x. If $y \in \chi$, prove that $d(x_n, y)$ converges to $d(x, y)$.

The triangle in equality gives

$$d(x,y) - d(x_n,y) \le d(x,x_n) \text{ and } d(x_n,y) - d(x,y) \le d(x_n,x).$$

In extremely elementary questions like this it is a good idea to provide all details, including the fact that $d(x_n, x) = d(x, x_n)$.

2. Let K be a closed subset and F be a closed subset in the metric space χ. If $K \cap F = \emptyset$, is it necessarily true that

$$0 < \inf\{d(x,y) : x \in K, y \in F\}?$$

As you all know, the answer to this is no, but the answer is yes if either F or K is compact. The simplest counterexample turned in was $F = \{n = 1, 2, 3, ..., n, ...\}$ and $K = \{2 + 1/2, 3 + 1/3, .., n + 1/n, ...\}$ as subsets of $\mathbb{R}$.

3. Are there infinite compact subsets of $\mathbb{Q}$?

Everyone's favorite example: $\{0\} \cup \{1/n,\ n = 1, 2, 3, ...\}$, considered as a subset of $\mathbb{R}$. That is easily seen to be closed (its complement is open) and bounded.

4. Prove that the space of continuous functions on the closed interval $[0, 1]$ with the metric

$$d(f,g) = \sup_{x \in [0,1]} |f(x) - g(x)|$$

is a complete metric space.

As stated this asks you to show that d is a metric, as well as showing completeness. I will skip showing that $d(x, y)$ is a metric.

Suppose $\{f_n\}$ is a Cauchy sequence in the metric d. Then, given $\epsilon > 0$ there is an $N(\epsilon)$ such that $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in [0, 1]$ when $n, m \ge N(\epsilon)$. Therefore for each $x \in [0, 1]$ the sequence $\{f_n(x)\}$ is a Cauchy sequence in $\mathbb{R}$ which converges to a value $f(x)$ by the completeness of $\mathbb{R}$. Since $|f_n(x) - f_m(x)| < \epsilon$ for all $x \in [0, 1]$ when $n, m \ge N(\epsilon)$, letting $m \to \infty$, we see $|f_n(x) - f(x)| \le \epsilon$ for all $x \in [0, 1]$ when $n \ge N(\epsilon)$. Thus $d(f_n, f) \le \epsilon$ when $n \ge N(\epsilon)$.

To complete this problem you should also show that f is continuous, but I'll skip that, too.

5. Consider the set of real-valued functions on $[0, 1]$ such that

$$|f(x) - f(y)| \le |x - y| \text{ and } \int_0^1 f(x)dx = 1.$$

Is this a compact subset of the metric space in problem 4?

Yes, the set, call it S, is certainly compact. To prove that, Arzela-Ascoli says it is enough to prove that S is an equi-continuous family of functions (that follows immediately from $|f(x) - f(y)| \leq |x - y|$) which is bounded and closed. To see that S is bounded note that if $f(x_0) > 2$ for some $x_0 \in [0, 1]$, then $f(x) > 1$ for all $x \in [0, 1]$. Likewise, if $f(x_0) < 0$, then $f(x) < 1$ for all $x \in [0, 1]$. So in either case $\int_0^1 f(x)dx \neq 1$. So $d(f, 0) \leq 2$ for all $f \in S$.

To see that S is closed, note that, since convergence in the metric on S implies uniform convergence $d(f_n, f) \to 0$ implies $\int_0^1 f_n(x)dx \to \int_0^1 f(x)dx$. So $\{f_n\} \in S$ implies $\int_0^1 f(x)dx = 1$. Also, when $d(f_n, f) < \epsilon$, we have for all $x, y \in [0, 1]$

$$|f(x)-f(y)| \leq |f(x)-f_n(x)|+|f_n(x)-f_n(y)|+|f_n(y)-f(y)| < \epsilon+|x-y|+\epsilon. \quad (1)$$

For any $\epsilon > 0$ convergence implies that there is an $N(\epsilon)$ such that $d(f_n, f) < \epsilon$ when $n \geq N(\epsilon)$. So (1) holds for all $\epsilon > 0$, and we have $|f(x) - f(y)| \leq |x - y|$. Thus S is closed.

A few people forgot that for general metric spaces compactness is equivalent to being complete and *totally* bounded. For closed subsets of $\mathbb{R}^n$ just being bounded is enough. It is *not* enough in this metric space, as the sequence $f_n(x) = \sin(nx)$ shows. That has no uniformly convergent subsequence.

6. Let $\{\mathcal{O}_j\}_{j=1}^n$ be a finite open cover of a compact metric space χ.

a) Show that there is an $\epsilon > 0$ such that for every $x \in \chi$ the ball $B(x, \epsilon)$ is contained in at least one of the $\mathcal{O}_j$'s.

There were both Heine-Borel and Bolzano-Weierstrass ways of doing this:

Bolzano-Weierstrass. Suppose there is no ϵ. Then we have $x_n \in \chi,\ n = 1, 2, \ldots$ such that $B(x_n, 1/n) \cap \mathcal{O}_j^c \neq \emptyset$ for all j. Since χ is compact, we have a subsequence x_{n_k} converging to $x_\infty \in \chi$. Since the $\mathcal{O}_j$'s cover χ and are open, there is an $\epsilon > 0$ such that $B(x_\infty, \epsilon) \subset \mathcal{O}_j$ for some j. However, this means that $B(x_{n_k}, 1/n_k) \subset \mathcal{O}_j$ when k is large enough that $\max\{d(x_{n_k}, x_\infty), 1/n_k\} < \epsilon/2$, contradicting the choice of the x_n's.

Heine-Borel. Since the $\{\mathcal{O}_j\}_{j=1}^N$ is an open cover of χ, for each $x \in \chi$ there is an $\epsilon(x)$ such that $B(x, 2\epsilon(x)) \subset \mathcal{O}_j$ for some j. Since χ is compact, a finite subset of the balls $B(x, \epsilon(x))$ covers χ. So suppose these ball have centers at $x_m,\ m = 1, .., N$ and let $\epsilon = \min\{\epsilon(x_m),\ m = 1, .., N\}$. Then given $y \in \chi$ we have

$$B(y, \epsilon) \subset B(x_m, 2\epsilon(x_m) \subset \mathcal{O}_j$$

for some choice of m and j.

b) Show that if none of the $\mathcal{O}_j$'s equals χ, then there is a largest ϵ with the property in part a).

Probably because it is an odd sort of question, this one caused you some difficulty. You needed to show that the set of ϵ's that will work is bounded, and that the supremum of that set was in the set. The easiest way that anyone found to show that the set is bounded was this. For each $\mathcal{O}_j$ there is an $x_j \in \chi$ such that $x \notin \mathcal{O}_j$. Given $x \in \chi$, let $D = \max\{d(x, x_j),\ j = 1, .., N\}$. Then $B(x, D + 1)$ cannot be contained in any $\mathcal{O}_j$, so all ϵ's that work must be less than $D + 1$. To see that the supremum, ϵ_0, of the set of admissible ϵ's is admissible, choose a sequence of admissible ϵ's increasing to it: then for any $x \in \chi$, $B(x, \epsilon_n) \subset \mathcal{O}_j$ for some j, and, since there are only a finite number of $\mathcal{O}_j$'s, there is a j_0 so that $B(x, \epsilon_{n_k}) \subset \mathcal{O}_{j_0}$ for a subsequence. Since $B(x, \epsilon_0) = \cup_k B(x, \epsilon_{n_k})$, we have $B(x, \epsilon_0) \subset \mathcal{O}_{j_0}$. Thus ϵ_0 is admissible.