Root Test, Ratio Test and Decreasing Sequences

Here are the facts in a nutshell.

Theorem 1. The series $\Sigma_{n=1}^{\infty} a_n$ is convergent if $\overline{\lim}\ |a_n|^{1/n} < 1$, divergent if $\overline{\lim}\ |a_n|^{1/n} > 1$, and can be either convergent or divergent if $\overline{\lim}\ |a_n|^{1/n} = 1$.

The proof of Theorem 1 is a direct application of the following: if $\overline{\lim}\ b_n = L < \infty$, then, for any $\epsilon > 0$, $b_n > L + \epsilon$ for only finitely many n, but $b_n > L - \epsilon$ for infinitely many n.

Theorem 2. If $a_n \neq 0$ for n sufficiently large, then one has

$$\underline{\lim}\ |\frac{a_{n+1}}{a_n}| \leq \underline{\lim}\ |a_n|^{1/n} \leq \overline{\lim}\ |a_n|^{1/n} \leq \overline{\lim}\ |\frac{a_{n+1}}{a_n}|.$$

The proof of Theorem 2 uses the property of $\overline{\lim}$ used in proving Theorem 1, the corresponding property of $\underline{\lim}$, and $\lim_{n\to\infty} p^{1/n} = 1$ for $p > 0$.

Corollary (to Theorem 1). Let $1/R = \overline{\lim}\ |a_n|^{1/n}$ with $R = 0$ if $\overline{\lim}\ |a_n|^{1/n} = \infty$ and $R = \infty$ if $\overline{\lim}\ |a_n|^{1/n} = 0$. Then the power series

$$\Sigma_{n=0}^{\infty} a_n x^n$$

converges for $|x| < R$ and diverges for $|x| > R$.

Corollary (to Theorems 1 and 2). $\Sigma_{n=1}^{\infty} a_n$ converges if $\overline{\lim}\ |\frac{a_{n+1}}{a_n}| < 1$ and diverges if $\underline{\lim}\ |\frac{a_{n+1}}{a_n}| > 1$.

Note that this corollary implies the "undergraduate ratio test": $\Sigma_{n=1}^{\infty} a_n$ converges if $\lim\ |\frac{a_{n+1}}{a_n}| < 1$ and diverges if $\lim\ |\frac{a_{n+1}}{a_n}| > 1$.

If a_n is a decreasing nonnegative sequence, i.e. $a_n \geq a_{n+1} \geq 0$, then one can check that if $n < 2^{m+1}$

$$\Sigma_{k=1}^{n} a_k \leq \Sigma_{k=1}^{m} 2^k a_{2^k}, \text{ and if } n \geq 2^m,\ \Sigma_{k=1}^{n} a_k \geq a_1 + \Sigma_{k=1}^{m} 2^{k-1} a_{2^k}.$$

From this one immediately deduces the handy convergence criterion (discovered by Cauchy):

Theorem 3 (Rudin, Theorem 3.27, and Tao, Prop.7.3.4). If $a_n \geq a_{n+1} \geq 0$, then Σa_n converges if and only if $\Sigma\ 2^n a_{2^n}$ converges.

For instance, $\Sigma\ n^{-p}$ converges if and only if the geometric series $\Sigma\ 2^{-(p-1)n}$ converges, i.e. if and only if $p > 1$, and $\Sigma (n(\ln n)^p)^{-1}$ converges if and only if the p-series $\Sigma\ (n \ln 2)^{-p}$ converges, i.e. if and only if $p > 1$.