

## Review of the Review

### Monday: Metric Spaces and Continuity

1. (Basic F'02#1) An easy one that has made several appearances on the basic.
2. Let  $d(x, E)$  be the function giving the distance from  $x$  to the set  $E$  in the metric space  $(X, d)$  (see formula (1) in the handout on B-W compactness implies H-B compactness). Note that

$$|d(x, E) - d(y, E)| \leq d(x, y).$$

Given two disjoint closed sets,  $A$  and  $B$  in  $X$ , let

$$\rho(x) = \frac{d(x, A)}{d(x, A) + d(x, B)}.$$

Show that  $\mathcal{O}_A = \{x \in X : \rho(x) < 1/3\}$  and  $\mathcal{O}_B = \{x \in X : \rho(x) > 2/3\}$  are disjoint open sets with  $A \subset \mathcal{O}_A$  and  $B \subset \mathcal{O}_B$

3. a) Suppose that  $f$  is a real-valued function on  $\mathbb{R}$  such that  $\lim_{h \rightarrow 0} (f(x+h) - f(x-h)) = 0$  for all  $x \in \mathbb{R}$ . Is  $f$  continuous on  $\mathbb{R}$ ?
- b) Is the function defined by

$$f(x, y) = \frac{4x^5y^2}{(x^4 + y^2)^2}, \quad (x, y) \neq (0, 0) \text{ and } f(0, 0) = 0$$

continuous at  $(0, 0)$ ?

### Tuesday: Riemann Integrals

1. (Basic S'05#4 in Analysis) Rated easy, given what we covered in the review.
2. In elementary calculus, when one is integrating continuous functions, one learns that Riemann sums converge to the integral as the mesh of the partition goes to zero (where the "mesh" is just the maximum length of an interval in the partition). Prove that this is true for any Riemann integrable function, too.
3. (Basic F'05#3) This one seems to be enormously popular with exam writers.

### Wednesday: Differentiation

1. (Basic F'02#6) Rated not so easy. This one mixes in a small amount of linear algebra.
2. Let  $f(x, y_1, y_2) = x^2y_1 + e^x + y_2$ . Show that there is a  $\delta > 0$  and a differentiable function  $g(y_1, y_2)$  defined on  $|y_1 - 1, y_2 - (-1)| < \delta$  such that  $g(1, -1) = 0$ ,  $f(g(y_1, y_2), y_1, y_2) = 0$ , and  $x = g(y_1, y_2)$  is the unique solution to  $f(x, y_1, y_2) = 0$  satisfying  $|x| < \delta$ .
3. (Rudin, p.244#31) Suppose that  $f(x, y)$  has continuous partial derivatives up to (and including) order 3. If the gradient of  $f$  vanishes at  $(x_0, y_0)$  but not all second order derivatives of  $f$  are 0 at  $(x_0, y_0)$ , show how one can then determine from the Taylor polynomial of  $f$  at  $(x_0, y_0)$  (of degree 2) whether  $f$  has a local maximum, or a local minimum, or neither, at the  $(x_0, y_0)$ .