A convergent series which is not absolutely convergent can be rearranged to sum to any number you choose, but rearrangement never changes the sum of an absolutely convergent series.

Part I. To see that rearrangement of a convergent but not absolutely convergent series can lead to any desired sum one can argue as follows:

Given a sequence of real numbers $\{a_n\}$, we can introduce the sequences $\{a_n^+\}$ and $\{a_n^-\}$ defined by

$$a_n^+ = \begin{cases} a_n \text{ if } a_n > 0 \\ 0 \text{ if } a_n \leq 0 \end{cases} \quad \text{and } a_n^- = \begin{cases} -a_n \text{ if } a_n < 0 \\ 0 \text{ if } a_n \geq 0 \end{cases}$$

Note that $|a_n| = a_n^+ + a_n^-$ and $a_n = a_n^+ - a_n^-$.

Proposition 1. If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n|$ diverges, then $\sum_{n=1}^{\infty} a_n^+ = \sum_{n=1}^{\infty} a_n^- = \infty$.

Proof: Since $\sum_{n=1}^{N} |a_n| = \sum_{n=1}^{N} a_n^+ + \sum_{n=1}^{N} a_n^-$, we cannot have both $\sum_{n=1}^{N} a_n^+$ and $\sum_{n=1}^{N} a_n^-$ bounded as $N \to \infty$. However, since $\sum_{n=1}^{N} a_n = \sum_{n=1}^{N} a_n^+ - \sum_{n=1}^{N} a_n^-$, if either of $\sum_{n=1}^{N} a_n^+$ or $\sum_{n=1}^{N} a_n^-$ is unbounded as $N \to \infty$, the other must also be unbounded.

Theorem: Let $\{a_n\}_{n=1}^{\infty}$ be a sequence of nonzero real numbers. If $\sum_{n=1}^{\infty} a_n$ converges and $\sum_{n=1}^{\infty} |a_n| = \infty$, then, given any $\alpha \in \mathbb{R} \cup \{+\infty\} \cup \{-\infty\}$, we can rearrange the terms $a_1, a_2, \ldots$ so that the new series converges to α.

Proof: Let $b_1, b_2, \ldots$ be the numbers $a_1^+, a_2^+, \ldots$ in the same order but with the zeroes omitted, and let $c_1, c_2, \ldots$ be the numbers $a_1^-, a_2^-, \ldots$ in the same order but with the zeroes omitted. Then any series consisting of all the b's with plus signs and all the c's with minus signs will be a rearrangement of $\sum_{n=1}^{\infty} a_n$.

I will take $\alpha > 0$ first, and give the changes needed for $\alpha < 0$ and $\alpha = \pm\infty$ at the end.

To build a series converging to α proceed as follows. The new series will be $\sum_{n=1}^{\infty} a'_n$. Take $a'_1 = b_1,\ a'_2 = b_2, \ldots$ until the first n such that

$$b_1 + b_2 + \cdots + b_n > \alpha$$

(n might be 1, but that is OK). Call this n, n_1. Then take $a'_{n_1+1} = -c_1,\ a_{n_1+2} = -c_2, \ldots$ until the first n such that

$$b_1 + b_2 + \cdots + b_{n_1} - c_1 - c_2 - \cdots - c_n < \alpha.$$

Call this n, n_2. Next repeat the process, adding b's until the sum is greater than α and then subtracting c's until the sum is less than α, back and forth, forever.

The series you get will look like

$$b_1+\cdots+b_{n_1}-c_1-\cdots-c_{n_2}+b_{n_1+1}+\cdots+b_{n_1+n_3}-c_{n_2+1}-\cdots-c_{n_2+n_4}+b_{n_1+n_3+1}+\cdots$$

I claim that $\lim_{N\to\infty}\sum_{n=1}^{N} a'_n = \alpha$. First, since $\sum_{n=1}^{\infty} b_n = \sum_{n=1}^{\infty} c_n = \infty$, no matter how many b's or c's you have used the sum of those remaining is still ∞. That means that at each stage you will be able to get above α or below α as required, and the integers n_k with $n_k \geq 1$ will be defined for all k. Secondly, for

$$n_1 + n_2 + \cdots n_{k-1} \leq N < n_1 + n_2 + \cdots n_k$$

the difference between $\sum_{n=1}^{N} a'_n$ and α is bounded by $b_{n_1+n_2+\cdots+n_{k-1}}$ if k is even and $c_{n_2+n_4+\cdots+n_{k-1}}$ if k is odd. Since $\sum_{n=1}^{\infty} a_n$ converges, $\lim_{n\to\infty} a_n = 0$ and this implies $\lim_{n\to\infty} b_n = \lim_{n\to\infty} c_n = 0$. Thus, given $\epsilon > 0$, there is a $K(\epsilon)$ such that $|\sum_{n=1}^{N} a'_n - \alpha| < \epsilon$ when $N \geq n_1 + n_2 + \cdots + n_{k-1}$ with $k \geq K(\epsilon)$. This completes the proof for $\alpha > 0$ (I admit that the last step here is a bit of a leap, but I think that more words will not help you understand it).

If $\alpha < 0$, just begin with c's instead of b's. If $\alpha = +\infty$, add b's until the sum is greater than 1. Then subtract c's until the sum is less than 1. Then add b's until the sum is greater than 2. Then subtract c's until the sum is less than 2. Continue this way, going up 1 when you add b's each time. If $\alpha = -\infty$, start by subtracting c's until the sum is less than -1. Then add b's until the sum is greater than -1. Then subtract c's until the sum is less than -2, and so on.

Part II. To see that rearrangement does not change the sums of absolutely convergent series one can argue as follows:

Assume that $\Sigma_{n=1}^{\infty} a_n = S$ and that Σa_n is absolutely convergent. Then, given $\epsilon > 0$, we can choose $N_1(\epsilon)$ so that $|\Sigma_{k=1}^{n} a_k - S| < \epsilon/2$ for $n \geq N_1(\epsilon)$, and choose $N_2(\epsilon)$ so that $\Sigma_{k=m}^{n}|a_k| < \epsilon/2$ for $m, n \geq N_2(\epsilon)$. We can assume that N_2 is chosen to be greater than N_1. Let σ, mapping $\mathbb{N}$ one-to-one onto $\mathbb{N}$, be the rearrangement, and choose $N(\epsilon)$ so that

$$\{n \in \mathbb{N} : n \leq N_2(\epsilon)\} \subset \{\sigma(k) : k \leq N(\epsilon)\}.$$

Note that $N(\epsilon) \geq N_2(\epsilon)$. Now for $n \geq N(\epsilon)$ we have

$$|\Sigma_{k=1}^{n} a_{\sigma(k)} - S| \leq |\Sigma_{k=1}^{N_2(\epsilon)} a_k - S| + \Sigma_{\{\sigma(k):\sigma(k)>N_2(\epsilon),k\leq n\}}|a_{\sigma(k)}| < \epsilon/2 + \epsilon/2.$$

Thus

$$\lim_{n\to\infty} \Sigma_{k=1}^{n} a_{\sigma(k)} = S.$$