On the inverse boundary value problem for linear isotropic elasticity and Cauchy-Riemann systems

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ABSTRACT. As the title suggests, this article has two distinct parts. In the first part we describe recent work on recovering the internal structure of an inhomogeneous elastic body from boundary measurements. This is an inverse problem which remains tantalizingly unsolved, even though there is strong evidence that boundary measurements should determine the internal structure uniquely in the isotropic case. In the second part we give new proofs of existence of solutions to systems of equations of the form $\partial_t C = AC$, for $C$ in $\text{GL}(m, \mathbb{C})$. We call these "Cauchy-Riemann systems". Solutions of these equations, depending smoothly or analytically on parameters, turn up in the study of many inverse problems, including the inverse boundary value problem for linear isotropic elasticity. The first part is expository, but the second is rather technical because we consider a case where $\partial_t$ degenerates to a directional derivative on the boundary of the parameter domain.

§1. An elastic body is isotropic if its structure is locally rotationally invariant, and in this case its elastic properties depend on just two functions, $\lambda(x)$ and $\mu(x)$, the "Lamé parameters". These parameters make the equations of linear elasticity relatively uncomplicated, but $\lambda$ has no direct physical interpretation. The simplest physical parameters are Young’s modulus, $E(x)$, and Poisson’s ratio, $\sigma(x)$. Young’s modulus is the ratio of force to stretching along any axis – in strict analogy with the constant in Hooke’s law. To understand Poisson’s ratio one should think of a narrow column of the elastic material which is being stretched lengthwise. Poisson’s ratio is the decrease in the radius of the middle of the column divided by the increase in its length.

The relations between $(E, \sigma)$ and $(\lambda, \mu)$ are (see, for instance, [7])

$$E = \frac{\mu(3\lambda + 2\mu)}{\lambda + \mu} \quad \sigma = \frac{\lambda}{2(\lambda + \mu)} \quad \text{and} \quad \lambda = \frac{E\sigma}{(1+\sigma)(1-2\sigma)} \quad \mu = \frac{E}{2(1+\sigma)}.$$ 

In common materials $E$ varies a great deal, but $\sigma$ does not. One always has $0 < \sigma < 1/2$, and for most materials $\sigma$ is between $1/4$ – as in glass – and $1/3$ – as in copper.

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The data for the inverse problem will be the force fields corresponding to deformations of the surface of the body. We will assume that these are known for all possible deformations in the limiting case of infinitesimal forces and deformations. This “deformation-to-force” map is the Dirichlet-to-Neumann map Λ associated with the equations of elastic equilibrium in a fixed region, Ω, in R^3. The first general result on this problem was that Λ determines the boundary normal derivatives of all orders for λ and μ. This is due to Nakamura and Uhlmann [5], and it immediately implies that Λ determines μ and ν uniquely, if they are real-analytic functions on ∂Ω. It is derived by examining the symbol series of Λ as a pseudo-differential operator to all orders.

In this article we will discuss results that do not assume analyticity, primarily the following from [4]:

**Theorem 1.** Suppose two isotropic elastic bodies with Lamé parameters (λ_1(x), μ_1(x)) and (λ_2(x), μ_2(x)), respectively, occupy the same region Ω ⊂ R^3 and have equal Dirichlet-to-Neumann maps. Then either of equalities λ_1(x) = λ_2(x) in Ω or μ_1(x) = μ_2(x) in Ω implies (λ_1(x), μ_1(x)) = (λ_2(x), μ_2(x)) in Ω.

This result also holds if one replaces (λ_1, μ_1) and (λ_2, μ_2) by (E_1, σ_1) and (E_2, σ_2).

In view of the lack of variation of σ in real problems one could say that the best statement would be: if Λ_1 = Λ_2 and σ_1(x) = σ_2(x) for x ∈ Ω, then E_1(x) = E_2(x) for x ∈ Ω.

**Preliminaries**

We will use the notation v = (v^1, v^2, v^3) for v ∈ C^3 and subscripts for derivatives, i.e. v^k_j is ∂v^k/∂x_j.

The deformation w = (w^1, w^2, w^3) leaves the elastic body at equilibrium, if the system of equations Lw = 0 holds, where the components of Lw are given by

\[ (Lw)^k = \sum_{j=1}^{3} (\lambda w^j)^k + \sum_{j=1}^{3} (\mu w^k_j + w^j_k), \quad k = 1, 2, 3. \]

The Dirichlet-to-Neumann map for L in a closed, bounded domain with smooth boundary, Ω ⊂ R^3, is given by

\[ Λ(f)^k = \sum_{j=1}^{3} (\lambda w^j)^k + \sum_{j=1}^{3} (\mu w^k_j + w^j_k), \quad k = 1, 2, 3, \]

where w is the solution of the Dirichlet problem Lw = 0 in Ω, w = f on ∂Ω, and ν is the unit outer normal to ∂Ω. Since Λ(f) is the force field on ∂Ω induced by the infinitesimal deformation f, one can think of Λ as the “deformation-to-stress” map in this setting.

An application of the divergence theorem from calculus leads to the following equivalent formulation of the inverse problem. Two elastic bodies in the region Ω with Lamé parameters (λ(1), μ(1)) and (λ(2), μ(2)), respectively, will have the same Dirichlet-to-Neumann maps if and only if

\[ 0 = H(w^{(1)}, w^{(2)}) = \text{def.} \int_Ω [(\lambda^{(2)} - \lambda^{(1)})(\nabla \cdot w^{(2)})](\nabla \cdot w^{(1)}) + \frac{1}{2} (\mu^{(2)} - \mu^{(1)}) \sum_{1 \leq j, k \leq 3} (w^{(2), j}_k + w^{(2), k}_j)(w^{(1), j}_k + w^{(1), k}_j) \] dx, \quad i = 1, 2
for all pairs \((w^{(1)}, w^{(2)})\) satisfying \(L^{(i)}w^{(i)} = 0\) in \(\Omega\).

Our overall strategy, as in a great many inverse boundary value problems, is to try to find enough solutions \(w^{(1)}\) and \(w^{(2)}\) so that the equations \(H(w^{(1)}, w^{(2)}) = 0\) will imply \(\lambda^{(1)} = \lambda^{(2)}\) and \(\mu^{(1)} = \mu^{(2)}\). The construction of solutions to \(Lw = 0\) is made much easier by the following result of Ang, Ikehata, Trong and Yamamoto[1]: there is a 4\(\times\)4 matrix \(V_0(\mathbf{x})\) such that

\[
w = \mu^{-1/2}u + \mu^{-1}\nabla f - f\nabla \mu^{-1}
\]

will satisfy \(Lw = 0\) when \(z = (u, f) \in \mathbb{C}^3 \times \mathbb{C}\) satisfies

\[
0 = Mz = \text{def.} \Delta \left( \begin{array}{c} u \\ f \end{array} \right) + V_1(x) \left( \begin{array}{c} \nabla f \\ \nabla \cdot u \end{array} \right) + V_0(x) \left( \begin{array}{c} u \\ f \end{array} \right)
\]

where

\[
V_1(x) = \begin{pmatrix} -2\mu^{1/2}\nabla^2 \mu^{-1} & -\mu^{-1}\nabla \mu \\ 0 & \frac{\lambda + \mu}{\lambda + 2\mu}\mu^{1/2} \end{pmatrix}
\]

and \(\nabla^2 f\) denotes the Hessian matrix \(\partial^2 f / \partial x_i \partial x_j\). We will not need the explicit form of \(V_0\) here. This system has the useful properties: a) the leading term is the scalar Laplacian, b) it is only 4 \(\times\) 4, and c) the matrix \(V_1\) is rather simple.

**Complex Exponential Solutions**

Since the leading term in \(M\) is \(\Delta\), we will assume that \(z = (u, f) = (e^{ix\cdot \mathbf{x}}, e^{ix\cdot \mathbf{s}})\), where \(\zeta \in \mathbb{C}^3\) satisfies \(0 = \zeta \cdot \zeta = \sum_{j=1}^{n} \bar{\zeta}_j^2\). In order to work in \(\mathbb{R}^3\) instead of \(\Omega\) the coefficients \(\lambda\) and \(\mu\) will be extended to be smooth, positive and constant for \(|x|\) large, and the coefficient matrices \(V_1\) and \(V_0\) will be cut off to vanish outside a ball containing \(\Omega\). Given orthogonal vectors \(l, \alpha, \beta \in \mathbb{R}^3\) with \(|\alpha| = |\beta| = 1\), we will choose (here \(\zeta\) will be used for \(w^{(1)}\) and \(\bar{\zeta}\) will be used for \(w^{(2)}\))

\[
\zeta = \frac{1}{2}l + (\tau^2 - \frac{l \cdot l}{4})^{1/2} \alpha + i\tau \beta \quad \text{and} \quad \bar{\zeta} = -\frac{1}{2}l + (\tau^2 - \frac{l \cdot l}{4})^{1/2} \alpha - i\tau \beta.
\]

These choices lead to

\[
z \cdot \zeta = \bar{\zeta} \cdot \zeta = 0, \quad \zeta - \bar{\zeta} = l, \quad \text{and} \quad \lim_{\tau \to \infty} \tau^{-1} \zeta(\tau) = \lim_{\tau \to \infty} \tau^{-1} \bar{\zeta}(\tau) = \alpha + i\beta = \text{def.} \theta.
\]

The plan is to use the simplifications in the solutions as \(\tau \to \infty\) to extract information from \(H(w^{(1)}, w^{(2)}) = 0\). This idea goes back to Sylvester and Uhlmann [8], and it has become a standard technique.

In terms of \(v = (r, s)\) the equation \(Mz = 0\) takes the form

\[
0 = M_\zeta v = \text{def.} \Delta v + 2i\zeta \cdot \nabla v + iA \cdot \zeta v + Bv,
\]

where \(A\) is a vector of matrices constructed from \(V_1\) and \(B\) is a first order operator whose coefficients do not depend on \(\zeta\). We are going to look for \(v\) in the form

\[
v = v_0 + v_1 + \cdots + v_n + v_c,\]

where \(v_j\) is order \(\tau^{-j}\) and the error term \(v_c\) is order \(\tau^{-n-1}\). Since \(\zeta - \tau\theta\) is bounded in \(\tau\), it will suffice to solve the equations

\[
2i\theta \cdot \partial_x v_0 + iA \cdot \theta v_0 = 0,
\]

\[
2i\theta \cdot \partial_x v_j + iA \cdot \theta v_j = \frac{1}{\tau}(-M_\zeta v_{j-1} + 2i\theta \cdot \nabla v_{j-1} + iA \cdot \theta v_{j-1}) = \text{def.} g_{j-1}, \quad j = 1, \ldots, n.
\]
The equation for \( v_0 \) was studied in [2] in the matrix form
\[
2i\theta \cdot \partial_x C + i A \cdot \theta C = 0,
\]
where \( C \) is a square matrix. It was shown in [2] that the equation has matrix solutions \( C(x, \theta) \) depending smoothly on \( \theta \) which are invertible for \( x \in \{ |x| \leq R \} \). When \( A(x) \) is simply a vector-valued function, one can choose \( C(x) \) tending to 1 as \( |x| \to \infty \). However, when the components of \( A(x) \) are \( n \times n \) matrices, \( n \geq 2 \), there may not exist solutions of (1) tending to the identity matrix as \( |x| \to \infty \), and to solve (1) one must allow \( C(x, \theta) \) to grow polynomially in the variables \( (x \cdot \alpha, x \cdot \beta) \).

In part II of this article we give a new, simpler proof of the existence of \( C(x, \theta) \), and an example where there are no solutions of (1) tending to the identity as \( |x| \to \infty \). We also prove a more refined existence result in two dimensions (i.e. \( x \in \mathbb{R}^2 \)) which was used in [3].

Applying these results we can choose \( v_0(x) = C(x, \theta)g(\theta \cdot x) \), where \( g \) is any vector of polynomials (note that \( \theta \cdot \partial_x g(\theta \cdot x) = 0 \)). The inhomogeneous equations for \( v_j, j \geq 1 \), have explicit solutions in terms of solutions of (1):
\[
v_j = (2i)^{-1} C(x, \theta)(\theta \cdot \partial_x)^{-1}(C^{-1}(\cdot, \theta))\psi g_{j-1}
\]
where \( \psi \) is a cutoff supported in the set where \( C \) is invertible satisfying \( \psi = 1 \) on a neighborhood of \( \Omega \). The final correction term, \( v_c \), must satisfy
\[
M_\xi v_\epsilon = -\psi_1 M_\xi (v_0 + \cdots + v_n) = -\psi_1 M_\xi v_n = O(\tau^{-n})
\]
where \( \psi_1 = 1 \) on a neighborhood of \( \Omega \) and the support of \( \psi_1 \) is contained in the set where the cutoffs used in the construction of \( v_1, \ldots, v_n \) are equal to 1. The equation (2) is solved in the article of Eskin mentioned above, and here the estimates on the inhomogeneous term imply that \( v_c \) and all of its derivatives will be \( O(\tau^{-n-1}) \) on \( \Omega \).

The next step is to substitute these complex exponential solutions into \( 0 = H(w^{(1)}, w^{(2)}) \) and compute the asymptotics as \( \tau \to \infty \). This leads to a lengthy calculation. The result is
\[
0 = \lim_{\tau \to \infty} \tau^{-2} H(w^{(2)}, w^{(1)}) = \int_\Omega e^{\int_x \pi(x)}(R^{(2)}, \pi^{(2)})V(x, \theta)(R^{(1)}, \pi^{(1)})dx, \quad (A)
\]
where for \( j = 1, 2 \), \( w^{(j)} \) corresponds to \( v^{(j)} = (e^{\pi \cdot x} \pi^{(j)} \cdot \pi^{(j)}), R^{(j)} = \mu_1^{-1/2} \theta \cdot \theta^{(j)}, \)
\[
V = \left( \frac{\lambda_1 + \mu_1 - \lambda_2 - \mu_2}{2(\mu_2^{-1} - \mu_1^{-1})} \frac{\mu_1 \mu_2}{\lambda_1 + 2\mu_1 + 2\mu_2} \right) \left( \frac{2(\mu_2^{-1} - \mu_1^{-1})\theta \cdot \theta^{(j)}b_2}{2(\mu_2^{-1} - \mu_1^{-1})(b_1 a_1 + b_2 a_2)} \right),
\]
\[
a_j = (\theta \cdot \theta^{(j)})^{-1} \quad \text{and} \quad b_j = \frac{\mu_j a_j + \mu_j}{2 \lambda_j + 2\mu_j}.
\]
Since “boundary uniqueness” result of Nakamura and Uhlmann cited earlier implies that \( \lambda_1 - \lambda_2 \) and \( \mu_1 - \mu_2 \) must vanish to all orders on \( \partial \Omega \), we can assume that \( V \) has compact support in \( x \).

Note that, since \( l \) is an arbitrary real vector perpendicular to \( \theta \), the identity (A) implies that the integral of this integrand over any plane perpendicular to
\[
\text{Re}\{\theta\} \times \text{Im}\{\theta\} \text{ will be zero. Hence, setting } z = \theta \cdot x = x_1 + ix_2, \text{ and, recalling the construction of } (R^{(j)}, s^{(j)}), \text{ we have}
\]
\[
0 = \int_{\mathbb{R}^2} g_2(z) \hat{C}_2^*(x_1, x_2, x_+, \theta) V(x_1, x_2, x_+, \theta) \hat{C}_1(x_1, x_2, x_+, \theta) g_1(z) dx_1 dx_2,
\]
where the \(\hat{C}\)'s are solutions of the reduced system
\[
\theta \cdot \partial_x \hat{C} = \bar{A} \hat{C}, \text{ where } \bar{A}(x, \theta) = \begin{pmatrix} 0 & a \\ -b & 0 \end{pmatrix}
\]

Now we need to exploit the freedom to vary \(\theta\) in these identities. We define \(C(x, e\theta) = eC(x, \theta), e \in \mathbb{C}\) for solutions of \((1)\), and this becomes \(\bar{C}(x, e\theta) = \begin{pmatrix} 1 & 0 \\ 0 & c^{-1} \end{pmatrix} \bar{C}(x, \theta)\) for the \(\bar{C}\)'s. Given the standard basis for \(\mathbb{R}^3\), \(\hat{e}_1, \hat{e}_2,\) and \(\hat{e}_3 = \hat{e}_1 \times \hat{e}_2\), we define
\[
\xi(t) = \frac{1}{2} (t-t^{-1}) \hat{e}_1 + i \frac{1}{2} (t+t^{-1}) \hat{e}_2 + \hat{e}_3
\]
for \(t \in \mathbb{C}\setminus\{0\}.\) Since \(\xi(t) \cdot \xi(t) = 0\) and \(\text{Re}\{\xi(t)\} \neq 0,\) \(\xi(t)\) has the representation \(\xi(t) = z(\alpha + i\beta)\) with \(z > 0,\) and hence \(C(x, \xi(t))\) is well-defined. Now we can apply complex variables arguments from \(\S 5-6\) in \([2]\) in the variable \(t.\) The result of these arguments is the following: there is a matrix \(B'\) with compact support in \(x\) of the form
\[
B'(x, \theta) = \begin{pmatrix} 0 & b_1(x) \\ b_2(x) & b_2(x) \cdot \theta \end{pmatrix}
\]
such that
\[
\theta \cdot \partial_x B'(x, \theta) + (\bar{A}^{(2)})^* B'(x, \theta) + B'(x, \theta) \bar{A}^{(1)} = V(x, \theta), \quad (B)
\]
where the \(\bar{A}^{(j)}\)'s and \(V\) are the matrices depending only on the Lamé parameters and \(\theta\) that were introduced earlier. The identities \((A)\) and \((B)\) give the information that one can obtain from the terms of order \(\tau^2\) in \(H(w^{(1)}, w^{(2)}).\)

\textbf{Proof of Theorem 1}

This is based on the equation \((B).\) When one writes out the equations in \((B)\) as a system, they become
\[
0 - b_2 b_{21} - b_1 b_{22} = v_{11} \quad (a)
\]
\[
\theta \cdot \partial_x b_{12} - b_2 b_{22} \cdot \theta = v_{12} \quad (b)
\]
\[
\theta \cdot \partial_x b_{21} - b_2 b_{22} \cdot \theta = v_{21} \quad (c)
\]
\[
\theta \cdot \partial_x b_{22} + \alpha_2 b_{12} + \alpha_1 b_{21} = v_{22} \quad (d),
\]
where \(v_{ij}\) are the entries in \(V.\) Since the \(b_{ij}\)'s have compact support, one can exploit the following (well-known) lemma

\textbf{Lemma.} Suppose that \(\mu, \nu\) are orthogonal unit vectors in \(\mathbb{R}^3\) and \(s > 0.\) Then for \(u \in H^1(\mathbb{R}^3)\) with compact support one has
\[
\int_{\mathbb{R}^3} e^{2s|\xi|^2} |(\mu + i\nu) \cdot \nabla u|^2 dx \geq 4s \int_{\mathbb{R}^3} e^{2s|\xi|^2} |u|^2 dx.
\]

Introducing the norms
\[
\|f\|_2^2 = \int_{\Omega} |f|^2 e^{2s|\xi|^2} dx,
\]

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and exploiting the form of the $v_{ij}$, application of the lemma to (a)-(d) leads to the conclusion that for $s > s_0$, one has
\[ \|\lambda_1 + \mu_1 - \lambda_2 - \mu_2\|_s \leq \frac{C}{s}\|\mu_1 - \mu_2\|_s \]
with $C$ independent of $s$. This implies Theorem 1. To prove the analogous theorem for Young’s modulus and the Poisson ratio, one uses the same estimate. If $\sigma_1 = \sigma_2 = \sigma$ in $\Omega$, then the estimate above can be written
\[ \|((E_1 - E_2)(\frac{\sigma}{1 + \sigma})(1 - 2\sigma) + \frac{1}{2(1 + \sigma)}\|_s \leq \frac{C}{s}\|((E_1 - E_2)(\frac{1}{2(1 + \sigma)}\|_s. \]
Since the coefficients involving $\sigma$ are bounded and bounded away from zero on $\Omega$, one concludes that $E_1(x) = E_2(x)$ for all $x \in \Omega$.

It is possible to solve the equations (a)-(d) explicitly, and conclude that a solution with $b_{ij}$ of compact support exists if and only if for each $\theta = \alpha + i\beta$ with $\alpha, \beta \in \mathbb{R}$, $|\alpha| = |\beta|$, $\alpha \cdot \beta = 0$ one has
\[ b_1^{1/2}(\theta \cdot \partial_\theta)^2(b_1^{-1/2} - b_1(\theta \cdot \partial_\theta)^2(\mu_1^{-1}) = b_2^{1/2}(\theta \cdot \partial_\theta)^2(b_2^{-1/2} - b_2(\theta \cdot \partial_\theta)^2(\mu_2^{-1}). \]
This amounts to a system of five partial differential equations which must be satisfied by the four functions $\mu_1, \mu_2, \lambda_1$ and $\lambda_2$. Nonetheless these equations do not imply that $\mu_1 = \mu_2$ and $\lambda_1 = \lambda_2$: one can find solutions of the equations with $\mu_1$ and $\lambda_1$ constant and $\mu_2$ and $\lambda_2$ smooth functions of $|x|^2$ which equal $\mu_1$ and $\lambda_1$ respectively outside small compact sets.

**Consequences of Higher Order Asymptotics of $H(w^{(1)}, w^{(2)})$.**

Since complex exponential solutions can be constructed to any given order in $r$, we have
\[ H(w^{(1)}, w^{(2)}) = \sum_{n=-2}^{\infty} \tau^{-n} H_n, \]
where each $H_n$ must vanish when $L^{(1)}$ and $L^{(2)}$ have equal Dirichlet-to-Neumann maps. The calculation of the higher order terms is complicated, and one must consider their dependence on $l$. The formula corresponding to (A) for the term of order zero involves the expansion of $(r, s)$ up to order $2$ so that $s^{(j)}$ is now $(s_0^{(j)}, s_{-1}^{(j)}, s_{-2}^{(j)})$ and similarly for $r^{(j)}$. With this notation $H_0$ has the form
\[ H_0 = \int_{\Omega} e^{i\|x\|[\lambda_1 - \lambda_2] + D_2(s^{(1)}, s^{(2)}) + E_2(r^{(1)}, s^{(2)}) + F_2(r^{(1)}, r^{(2)}) + G_3(s^{(1)}, s^{(2)}) + G_2(r^{(1)}, r^{(2)}) + \bar{G}_3(r^{(2)}, s^{(1)}) + \bar{G}_2(r^{(1)}, r^{(2)})]dx, \]
where $D_{ij}, E_{ij}, etc.$, are Hermitian forms with coefficients which are symbols in $(x, l)$ of order $j$. Hence, $H_0 = 0$ corresponds to a pseudo-differential equation of the form
\[ P_3(x, D)(\mu_1 - \mu_2) + P_2(x, D)(\lambda_1 - \lambda_2) = 0, \]
but the symbols of the operators $P_1$ and $P_2$ are far from being explicit. We have been able to obtain useful information from $H_0 = 0$ only in the case that $\mu_1$ and $\mu_2$ are close to constants. In an early version of [4] we assumed that $\lambda_1$ and $\lambda_2$ were also close to constants and proved the following local uniqueness result:
Theorem 2. Given that $\Lambda_1 = \Lambda_2$, there is an $\epsilon > 0$ and a $k$ such that $|\lambda_j - \lambda_0|_{C^k(\Omega)} < \epsilon$ and $|\mu_j - \mu_0|_{C^k(\Omega)} < \epsilon$, $j = 1, 2$, for positive constants $\lambda_0$ and $\mu_0$ imply $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$ in $\Omega$.

Nakamura and Uhlmann then showed that the following “semi-local” uniqueness holds ([6]):

Theorem 3. Given that $\Lambda_1 = \Lambda$ and $\lambda_j$, $\mu_j$ and $\mu_j^{-1}$, $j = 1, 2$, belong to a bounded set, $B$, in $C^k(\Omega)$ for $k$ sufficiently large, there is an $\epsilon(B) > 0$ such that $|\nabla \mu_j|_{C^{k-1}(\Omega)} < \epsilon(B)$, $j = 1, 2$, implies $(\lambda_1, \mu_1) = (\lambda_2, \mu_2)$.

This also can be derived using $H_0$, and a proof is included in §5 of [4].

§II. This section is devoted to proofs of existence of solutions to equation (1). We are particularly concerned with existence of solutions with good dependence on parameters. The proofs are based on a lemma of H. Cartan (see, for example, [9] or [10]). We begin with the problem from §I, generalized to $n$ dimensions.

Let $G(2,n)$ be the manifold of all two-dimensional planes through the origin in $\mathbb{R}^n$, $n \geq 3$, and set $Y = G(2,n) \times [-2R, 2R]^{n-2}$. Let $B(y, x)$ be a $C^\infty$ function with support in $|x| < R$ from $Y \times \mathbb{R}^n$ to the $m \times m$ matrices. We identify $\mathbb{R}^2$ with the complex plane, setting $z = x_1 + ix_2$ and $\partial_z = \frac{1}{2}(\partial_{x_1} + i\partial_{x_2})$. The following theorem was proven in [2] and was used in [2] and [4].

Theorem 4. There exists a smooth function $C(y, x)$ on $Y \times \{|x| \leq 2R\}$ with values in $GL(m, \mathbb{C})$, satisfying

$$\partial_z C = B(y, x) C. \tag{3}$$

Note that this applies to the problem in §I: we set $x_1 = x \cdot \alpha$ and $x_2 = x \cdot \beta$. The matrix $A(x) \cdot \theta$ depends on $(\alpha, \beta)$, but, since we require $C(x, \exp(i\alpha)\theta) = C(x, \theta)$, we can treat this coefficient as a function of the plane spanned by $\alpha$ and $\beta$. Then we have $B(y, x_1, x_2) = A(x) \cdot \theta$ modulo multiplication by $\exp(i\alpha)$ and $Y = G(2,3) \times [-2R, 2R]$, where the final component of $y$ is the coordinate $x^+ = x - x_1\alpha - x_2\beta$.

In [2] the solution $C(y, x)$ was defined on $Y \times \mathbb{C}$ with polynomial growth in $x$. However, for the applications in [2] and [4] it suffices to have (3) hold for $C$ defined on $Y \times \{|x| < 2R\}$

Proof. Choose $\phi(x) \in C^\infty(\mathbb{R}^2)$ satisfying $\phi(x) = 1$ for $|x| < 1$ and $\phi(x) = 0$ for $|x| > 2$. For an arbitrary $x_0 \in \mathbb{R}^2$ set $\phi_\epsilon(x) = \phi((x - x_0)/\epsilon)$, $\epsilon > 0$. Let $\Pi$ be the inverse of $\partial_z$, i.e.

$$[\Pi f](x) = \frac{2}{(2\pi)^2} \int_{\mathbb{R}^2} \frac{\hat{f}(\xi) e^{ix \cdot \xi}}{\xi_1 + i\xi_2} d\xi_1 d\xi_2 = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{f(w)}{w_1 - w_1 + i(x_2 - w_2)} d\omega_1 d\omega_2,$$

and consider the integral equation

$$C - \Pi \phi \epsilon BC = I \text{ in } \mathbb{R}^2. \tag{4}$$

Let $C_0(\mathbb{R}^2)$ be the space of continuous matrix-valued functions tending to zero as $|x| \to \infty$ with the supremum norm, i.e.

$$||f||_{\infty} = \sup_x ||f(x)||,$$
where \(|f(x)|| is the norm of \(f(x)\) as an operator on \(\mathbb{C}^n\). Then
\[
||\Pi \phi_x B h||_\infty \leq \frac{1}{\pi} \sup_x \int_{\mathbb{R}^2} \frac{|\phi_x(w)|}{\sqrt{(x_1 - w_1)^2 + (x_2 - w_2)^2}} \, dw_1 \, dw_2 \sup_{y, x} ||B(y, x)|| \cdot ||h||_\infty.
\]
Therefore the norm of the operator \(\Pi \phi_x B\) is small uniformly on \(Y\) if either \(\epsilon\) or \(\sup_{y, x \in \mathbb{R}^2} ||B(y, x)||\) is small.

We will solve (4) with \(C = I + G\). Hence we require
\[
G - \Pi \phi_x B G = \Pi \phi_x B.
\]
(5)
We can take \(\epsilon\) small enough that the operator norm of \(\Pi \phi_x B\) is less than \(1/2\) and \(||\Pi \phi_x B||_\infty < 1/4\). Hence \(I - \Pi \phi_x B\) is an invertible operator on \(C_0(\mathbb{R}^2)\) with norm at most 2, and the solution \(G\) of (5) exists with \(||G||_\infty < 1/2\). Moreover, \(G\) is continuous in \(y\) since \(\Pi \phi_x B(y, \cdot)\) is norm continuous in \(y\). Differentiating (5) with respect to \(y\) and \(x\), one sees that \(G\) is actually \(C^\infty\) in \(y\) and \(x\) – though we have no control over the size of its derivatives. Since \(||G||_\infty < 1/2\), it follows that \(C = I + G\) is invertible for all \((y, x) \in Y \times \{|x| \leq 2R\}\). Hence, applying \(\partial_x\) to both sides of (4), we see that \(C(x, y)\) is a smooth invertible solution to
\[
\partial_x C(y, x) = B(y, x)C(y, x)
\]
on \(Y \times \{|x - x_0| < \epsilon\}\) with \(\epsilon\) independent of \(x_0\).

Let \(\{U_j\}_{j=1}^N\) be a finite cover of \(\{|x| \leq 2R\}\) by open squares of side length \(\epsilon\), chosen so that for each \(k, 1 \leq k < N, \cup_{j=1}^k U_j\) and \(\cup_{j=1}^k U_j \cap \cup_{k+1}^N U_j\) are simply connected. On each \(U_j\) we have an invertible solution of (3), \(C_j \in C^\infty(Y \times U_j)\). Note that \(C_j^{-1} C_j\) satisfies
\[
\partial_x C_j^{-1} C_j = (\partial_x C_j^{-1}) C_j + C_j^{-1} \partial_x C_j = -C_j^{-1} BC_j + C_j^{-1} BC_j = 0
\]
on \(U_j \cap U_j\), and hence is holomorphic on \(U_j \cap U_j\).

To complete the construction of a solution \(C\) of (3) on \(Y \times \{|x| \leq 2R\}\) we will apply the following lemma of H. Cartan (see Malgrange [9] or Gunning and Rossi [16]).

**Cartan’s Lemma.** Suppose \(O_1, O_2\) and \(O_1 \cap O_2\) are simply connected sets in \(\mathbb{C}\) and \(h\) is a holomorphic function on \(O_1 \cap O_2\) with values in \(GL(m, \mathbb{C})\). Then \(h\) can be factored as
\[
h = h_1 h_2^{-1}
\]
where \(h_j\) is a holomorphic function on \(O_j, j = 1, 2,\) with values in \(GL(n, \mathbb{C})\). Moreover, this factorization preserves smooth or analytic dependence of \(h\) on parameters.

To apply Cartan’s Lemma here we begin with \(O_1 = U_1, O_2 = U_2\) and \(h = C_1^{-1} C_2\). Then the lemma gives us
\[
\tilde{C}_2 = \begin{cases} C_1 h_1 \text{ in } U_1 \\ C_2 h_2 \text{ in } U_2 \end{cases}
\]
Then we apply Cartan’s Lemma with \(O_1 = U_1 \cup U_2, O_2 = U_3\) and \(h = \tilde{C}_2^{-1} C_3\). Continuing in this way, we arrive after \(N\) steps at a function \(C\) satisfying the requirements of Theorem 4, and complete this proof.

In [3] one needs to solve the equation (1) in the form
\[
\zeta(t) \cdot \partial_x C = A(x) \cdot \zeta(t) C,
\]
(6)
where \(x \in \mathbb{R}^2, t \in \mathbb{C}, \zeta(t) = \frac{1}{2}(t + \frac{1}{t}, i(t - \frac{1}{t}))\), and \(A(x)\) is a smooth matrix function supported in \(|x| < R\). Solutions of (6), \(C(x, t) \in GL(m, \mathbb{C})\), are required in the
domains \( D_+ = \{ (x, t) : |x| \leq 2R, \ |t| \leq 1 \} \) and \( D_- = \{ (x, t) : |x| \leq 2R, \ |t| \geq 1 \} \). Moreover, these solutions must be analytic in \( t \) for \( |t| < 1 \) on \( D_+ \) and \( |t| > 1 \) (including \( t = \infty \)) on \( D_- \), and continuous up to \( |t| = 1 \). This is used in [3] to set up a Riemann-Hilbert problem. Thus the result which is needed can be formulated as follows.

**Theorem 5.** There are solutions \( C_+(x, t) \) and \( C_-(x, t) \) on \( D_+ \) and \( D_- \) respectively with values in \( GL(m, \mathbb{C}) \) analytic in \( |t| < 1 \) and \( |t| > 1 \) with continuous extensions to \( |t| = 1 \).

The outline of the proof of Theorem 5 is the same as that of Theorem 4. However, for \( t = \exp(i\theta) \) the differential operator in (6) degenerates to the directional derivative \( \cos \theta \partial_{x_1} - \sin \theta \partial_{x_2} \), and the inverse \( \Pi(t) \) of \( \zeta(t) \cdot \partial_t \) includes the Hilbert transform with respect to the variable \( x_1 \sin \theta + x_2 \cos \theta \). Since the Hilbert transform is not bounded in the supremum norm, we will work in H"older spaces: let \( C^{1/2}(\mathbb{R}^2) \) be the space of H"older-1/2 matrix functions with the norm

\[
\|f\|_{1/2} = \sup_x \|f(x)\| + \sup_{|x-x'| \leq 1} \frac{\|f(x) - f(x')\|}{|x-x'|^{1/2}},
\]

where, and before, \( \|f(x)\| \) denotes the norm of \( f(x) \) as an operator on \( \mathbb{C}^n \). The use of \( C^{1/2} \) complicates the proof. In particular, we need the following lemma.

**Lemma 1.** Let \( \phi(x) = \phi(|x|) \in C^\infty_c(\{ |x| < 2 \}) \) satisfy \( \phi(x) = 1 \) for \( |x| \leq 1 \), and set \( \phi_e(x) = \phi(x/e) \). Then the operator \( t^{-1} \Pi(t) \phi_e \) is bounded on the space \( C^{1/2}(\mathbb{R}^2) \) and its norm goes to zero as \( e \) goes to zero, uniformly on the disk \( \{ |t| \leq 1 \} \). Likewise the operator \( t \Pi(t) \phi_e \) is bounded on \( C^{1/2}(\mathbb{R}^2) \) with norm going to zero uniformly on \( \{ |t| \geq 1 \} \).

**Proof of Lemma 1.** We will give this proof for the \( |t| \leq 1 \) case. The proof for \( |t| \geq 1 \) is the same. Writing \( t = re^{i\theta} \), we have

\[
\zeta(t) = \frac{1}{2} (r + \frac{1}{r}) (\cos \theta, -\sin \theta) + \frac{i}{2} (r - \frac{1}{r}) (\sin \theta, \cos \theta).
\]

Hence, after rotating coordinates, the equation \( \zeta(t) \cdot \partial_t u = f \), becomes

\[
a \frac{2r}{r} \partial_{x_1} u + \frac{i}{r} \partial_{x_2} u = \tilde{f}, \quad \text{where} \quad \tilde{u}(x) = u(Rx), \quad \tilde{f}(x) = f(Rx),
\]

\[
R = \begin{pmatrix} \cos \theta & \sin \theta \\ -\sin \theta & \cos \theta \end{pmatrix}, \quad a = r^2 + 1 \quad \text{and} \quad b = r^2 - 1.
\]

Since \( \Pi(t) \) is the fundamental solution for \( \zeta(t) \cdot \partial_t u = f \) obtained by taking the inverse Fourier transform of \( \langle i\zeta(t) \cdot \xi \rangle^{-1} \), in these coordinates we have

\[
[r^{-1} \Pi(t) \phi_e f](x) = \frac{1}{\pi} \int_{\mathbb{R}^2} \frac{\phi_e(y) f(y)}{b(x_1 - y_1) + ia(x_2 - y_2)} dy_1 dy_2 = \frac{1}{\pi} \int_{|x_2 - y_2| < \sqrt{\tau}} \frac{\phi_e(y_1, x_2) f(y_1, x_2)}{b(x_1 - y_1) + ia(x_2 - y_2)} dy_1 dy_2
\]

\[+ \frac{1}{\pi} \int_{|x_2 - y_2| > \sqrt{\tau}} \frac{\phi_e(y) f(y)}{b(x_1 - y_1) + ia(x_2 - y_2)} dy_1 dy_2 = \text{def.} \ I_1(x) + I_2(x) + I_3(x).
\]
This proof consists of estimating the norms of $I_j$, $j = 1, 2, 3$ in $C^{1/2} (\mathbb{R}^2)$. This is elementary but a little tedious.

Doing the integration in $y_2$ in $I_1$ one finds

$$I_1(x) = \frac{2}{\pi a} \int_{-\infty}^{\infty} \phi_\epsilon(y_1, x_2)f(y_1, x_2) \operatorname{Arctan} \left( \frac{a \sqrt{\epsilon}}{b(x_1 - y)} \right) dy_1.$$  

Hence $\sup |I_1(x)| \le C \epsilon \sup |f(x)|$. To estimate difference quotients we set $\Delta x = x - x'$, and

$$I_1(x) - I_1(x') = [I_1(x_1, x_2) - I(x_1', x_2)] + [I_1(x_1', x_2) - I_1(x_1', x_2')]$$

Let $I_{11}(x, x') + I_{12}(x', x_2)$

Making the change of variables $y_1 \to y_1 - \Delta x_1$ in the second term in $I_{11}$, we have

$$I_{11} = \frac{2}{\pi a} \int_{-\infty}^{\infty} \left[ \phi_\epsilon(y_1, x_2) f(y_1, x_2) - \phi_\epsilon(y_1 - \Delta x_1, x_2) f(y_1 - \Delta x_1, x_2) \right]$$

$$\times \operatorname{Arctan} \left( \frac{a \sqrt{\epsilon}}{b(x_1 - y_1)} \right) dy_1.$$  

Hence, considering the cases $|\Delta x| < \epsilon$ and $|\Delta x| > \epsilon$ separately, we see

$$\sup \frac{|I_{11}(x, x')|}{|\Delta x|^{1/2}} \le C(\epsilon \sup |f(x)| + \epsilon |f|_{1/2})$$

where $|f|_{1/2} = \sup |\Delta x|^{-1/2} |f(x + \Delta x) - f(x)|$. For $I_{12}$ one needs no change of variables to arrive at the same estimate. Thus we have the necessary estimates for $I_1$.

For $I_2$ we have

$$\le \frac{1}{\pi a} \int_{|x_2 - y_2| < \sqrt{\epsilon}} \frac{|I_2(x)|}{|x_2 - y_2|} dy.$$

This implies

$$\sup |I_2(x)| \le C(\epsilon^{1/2} \sup |f| + \epsilon^{3/2} |f|_{1/2}).$$

The most difficult term to estimate is

$$I_2(x) - I_2(x') = [I_2(x) - I_2(x_1', x_2)] + [I_2(x_1', x_2) - I_2(x')]$$

Let $I_{21}(x, x_1') + I_{22}(x', x_2)$.

For this we will use the following result from calculus. If $f \in C^2 (\mathbb{R}^2)$ satisfies $f(s, 0) = f(0, t) = 0$ for all $s$ and $t$, then $f = stg(s, t)$, where $|g|$ is bounded by the $C^2$-norm of $f$. One proves this by writing

$$f(s, t) = ts \int_0^1 \int_0^1 g(\alpha s, \beta t) d\alpha d\beta.$$  

Changing variables $y_1 \to y_1 - \Delta x_1$, we have

$$I_{21} = \frac{1}{\pi} \int_{|x_2 - y_2| < \sqrt{\epsilon}} \frac{(\phi_\epsilon(y) - \phi_\epsilon(y_1, x_2))(f(y_1 - \Delta x_1, y_2) - f(y))}{b(x_1 - y_1) + ia(x_2 - y_2)} dy +$$

$$\int_{|x_2 - y_2| < \sqrt{\epsilon}} \frac{(\phi_\epsilon(y_1 - \Delta x_1, y_2) - \phi_\epsilon(y) - \phi_\epsilon(y_1 - \Delta x_1, x_2) + \phi_\epsilon(y_1, x_2))(f(y - \Delta x))}{\pi(b(x_1 - y_1) + ia(x_2 - y_2))} dy.$$  

Hence we have

$$\frac{|I_{21}|}{|\Delta x|^{1/2}} \le C \epsilon^{1/2} |f|_{1/2} +$$  

(8)
\[
\frac{1}{a\pi} \int_{|x_2 - y_2| < \sqrt{\tau}} \frac{[\phi_1(y_1 - \Delta x_1, y_2) - \phi_1(y_1, y_2) - \phi_1(y_1 - \Delta x_1, x_2) + \phi_1(y_1, x_2)]}{|\Delta x|^{1/2}} dy \sup |f|
\]

Applying (7) in the variables \( s = x_2 - y_2 \) and \( t = x_1 - x'_1 \), we have

\[
\phi_1(y_1 - \Delta x_1, y_2) - \phi_1(y_1, y_2) - \phi_1(y_1 - \Delta x_1, x_2) + \phi_1(y_1, x_2) = (x_1 - x'_1)(x_2 - y_2) h(x, x') , y)
\]

where \( |h| \leq C e^{-2\epsilon} \). Thus, splitting the integral in (8) into integration over \( |\Delta x|^{1/2}|x_2 - y_2| > \delta \) and \( |\Delta x|^{1/2}|x_2 - y_2| < \delta \), and using (9) in the second region, we have

\[
\frac{|I_{21}|}{|\Delta x|^{1/2}} \leq C(\epsilon^{1/2}|f|_{1/2} + \frac{\epsilon^{3/2}}{\delta} + \delta) \sup |f|.
\]

Taking \( \delta = \epsilon^{3/4} \), we conclude

\[
\frac{|I_{21}|}{|\Delta x|^{1/2}} \leq C(\epsilon^{1/2}|f|_{1/2} + \epsilon^{1/4} \sup |f|).
\]

The same argument, carried out in the second variable, gives the same estimate for \( I_{22} \), and we have the necessary estimates for \( I_2 \).

For \( I_3 \) we have immediately \( |I_3(x)| \leq C \epsilon^{3/2} \sup |f| \), and, making the change of variables \( y 
\rightarrow y - \Delta x \) gives

\[
I_3(x) - I_3(x') = \int_{|x_2 - y_2| > \sqrt{\tau}} \frac{(\phi_1(y) - \phi_1(y - \Delta x)) f(y) + \phi_1(y - \Delta x)(f(y) - f(y - \Delta x))}{b(x_1 - y_1) + ia(x_2 - y_2)} dy.
\]

From this, considering the cases \( |\Delta x| < \epsilon \) and \( |\Delta x| > \epsilon \) separately, we see

\[
\frac{|I_3(x) - I_3(x')|}{|\Delta x|^{1/2}} \leq C(\epsilon \sup |f| + \epsilon^{3/2} |f|_{1/2}).
\]

This gives the necessary estimate for \( I_3 \), and completes the proof of the lemma.

**Proof of Theorem 5.** The estimates in Lemma 1 are clearly unchanged, if, instead of using \( \phi_1(x) = \phi(x)/\epsilon \), we set \( \phi_1(x) = \phi((x - x_0)/\epsilon) \) for an arbitrary \( x_0 \in \mathbb{R}^2 \). We will assume the latter form for \( \phi_1 \) here on. Consider the integral equation

\[
C(t) - \Pi(t)\phi_A \cdot \zeta(t) C(t) = I.
\]

Lemma 1 implies

\[
||\Pi(t)\phi_A \cdot \zeta(t)||_{1/2} \leq C(\epsilon)||tA \cdot \zeta(t)||_{1/2}||h||_{1/2},
\]

for \( |t| \leq 1 \) with \( C(\epsilon) \) independent of \( t \) and tending to zero as \( \epsilon \to 0 \). Therefore the norm of the operator \( \Pi(t)\phi_A \cdot \zeta(t) \) is small uniformly on \( \{ |t| \leq 1 \} \) if either \( \epsilon \) or \( ||tA \cdot \zeta(t)||_{1/2} \) is small.

Since \( x \in \mathbb{R}^2 \), the operator \( \Pi(t)\phi_A \) is complex analytic in \( t \) in \( 0 < |t| < 1 \). This follows from Lemma 6.1 in [2]. Since \( \Pi(t)\phi_A \cdot \zeta(t) = (t^{-1}\Pi(t)\phi_A)(tA \cdot \zeta(t)) \) is bounded near \( t = 0 \), the singularity at \( t = 0 \) is removable, and \( \Pi(t)\phi_A \cdot \zeta(t) \) is analytic on \( |t| < 1 \). Arguments similar to those in Lemma 1 show that \( \Pi(t)\phi_A \cdot \zeta(t) \) extends continuously to \( |t| \leq 1 \) as an operator-valued function, but we will omit this proof here.

We will solve (6) with \( C(t) = I + G(t) \). Hence we require

\[
G(t) - \Pi(t)\phi_A \cdot \zeta(t) G(t) = \Pi(t)\phi_A \cdot \zeta(t).
\]

As in the proof of Theorem 4, we can take \( \epsilon \) small enough that the operator norm of \( \Pi(t)\phi_A \cdot \zeta(t) \) is less than \( 1/2 \) for \( |t| \leq 1 \) and \( ||\Pi(t)\phi_A \cdot \zeta(t)||_{1/2} < 1/4 \) for \( |t| \leq 1 \).
Thus the solution of (11) exists with \( \|G(t)\|_{1/2} < 1/2 \) for \( |t| \leq 1 \). Moreover, since we can write
\[
G(t) = \sum_{j=1}^{\infty} (\Pi(t) \phi \cdot \zeta(t))^j,
\]
and this sum converges uniformly on \( |t| \leq 1 \), we may conclude that \( G(t) \) is analytic in \( |t| < 1 \) and continuous on \( |t| \leq 1 \).

Applying \( \zeta(t) \cdot \partial_t \) to both sides of (10), we have
\[
\zeta(t) \cdot \partial_t C = A(x) \cdot \zeta(t) C
\]
for \( (x, t) \in \{ |x - x_0| < \epsilon \} \times \{ |t| \leq 1 \} \). At this point there is a serious problem. Given solutions of (6), \( C_1(t) \) and \( C_2(t) \), defined on intersecting sets \( D_1 \) and \( D_2 \), let \( h(x, t) = C_1^{-1}(t) C_2(t) \). Then \( \zeta(t) \cdot \partial_t h = 0 \) on \( D_1 \cap D_2 \). In the proof of Cartan’s Lemma one approximates \( h \) on \( D_1 \cap D_2 \) by \( \hat{h} \) which satisfies \( \zeta(t) \cdot \partial_t \hat{h} = 0 \) on all of \( \{ |x| \leq 2R \} \). We have not found a way to construct such an \( \hat{h} \) so that it approximates \( h \) on \( D_1 \cap D_2 \) uniformly in \( t \) on \( \{ |t| \leq 1 \} \) and is analytic in \( t \) in \( \{ |t| < 1 \} \). That is what would be needed to use the standard proof of Cartan’s Lemma here. To get around this we will construct solutions to (6) on \( \{ |x| \leq 2R \} \times \{ |t| \leq 1 \} \) in two stages (labelled “Step 1” and “Step 2” in what follows). First, using the method of Cartan’s Lemma, we will construct \( C(t) \)'s on \( \{ |x| < 2R + 1 \} \times \{ |t - t_0| < \delta, |t| \leq 1 \} \) for each \( t_0 \) on \( |t| \leq 1 \). These local solutions in \( t \) will have the following additional property. Let \( \zeta^{-1}(t) = \frac{1}{2}(t^2 - 1, i(t^2 + 1)) \) (note that \( \zeta^{-1}(e^{i\theta}) = e^{i\theta}(\sin \theta, \cos \theta) \)). Given \( C_1(t) \) and \( C_2(t) \) defined on intersecting disks in \( t \), we have
\[
C_1^{-1}(t) C_2(t)[x] = f(\zeta^{-1}(t) \cdot x, t) \text{ when } |x| \leq 2R,
\]
where \( f(z, t) \) is analytic on \( \{ |z| \leq 2R \} \) and analytic on \( \{ |t - t_0^{(1)}| < \delta_1, |t| < 1 \} \cap \{ |t - t_0^{(2)}| < \delta_2, |t| < 1 \} \) with a continuous extension to the portion of \( \{ |t| = 1 \} \) in the boundary of this set. The point here is that the domain of analyticity of \( f \) is fixed and fairly large, although the range of \( \zeta^{-1}(t) \cdot x \) degenerates to a line as \( |t| \to 1 \). Next we will apply the method of Cartan’s Lemma a second time to construct a solution of (6) on all of \( \{ |x| \leq 2R \} \times \{ |t| \leq 1 \} \) from these local solutions in \( t \). The property in (12) will make this second application of Cartan’s Lemma essentially standard.

**Step 1.** We will begin with the first case mentioned above, i.e. the construction of solutions on \( \{ |x| < 2R + 1 \} \times \{ |t - t_0| < \delta, |t| \leq 1 \} \) with \( |t_0| = 1 \). Once one understands this case, the construction for the \( |t_0| < 1 \), will require only a little more explanation. Given \( t_0 = \exp(i\theta_0) \), we will begin by constructing solutions in strips \( \{ y : |y_1 - y_0| < \alpha \} \), where
\[
(y_1, y_2) = (x_1 \cos \theta_0 - x_2 \sin \theta_0, x_1 \sin \theta_0 + x_2 \cos \theta_0).
\]
We partition this strip into squares of side \( \alpha \), but, since \( A(x) \) is supported in \( \{ |y| < R \} \), we only cover the portion of the strip with \( |y_2| \leq R \) with these squares. Taking \( \alpha \) small enough that squares of side \( 3\alpha / 2 \) are contained in disks of radius \( \epsilon \), we have the solutions to \( \zeta(t) \cdot \partial_x C = \phi \cdot \zeta(t) \cdot AC \) from Lemma 1 with the support of \( \phi \) centered at the center of the squares. Hence \( C \) satisfies \( \zeta(t) \cdot \partial_x C = \zeta(t) \cdot AC \) in the square of side \( 3\alpha / 2 \). In what follows we will use constants \( \alpha' \) and \( \alpha'' \) satisfying \( \alpha < \alpha'' < \alpha' < 3\alpha / 2 \).

Since our setting is not the standard one, and we require additional properties in the solutions, we will go through the details of the proof of Cartan’s Lemma.
The essential argument is the following. Given $C_1$ and $C_2$, satisfying $\zeta(t) \cdot \partial_x C = A(x) \cdot \zeta(t) C$ on overlapping rectangles $R_i$, $i = 1, 2$,

$$R_i = \{ y : |y_1 - y_0| < \alpha', \ a_i < y_2 < b_i \} \text{ with } a_1 < a_2 < b_1 < b_2,$$

we define $h = C_1^{-1} C_2$ on $R_1 \cap R_2$. The first step is to approximate $h$ on $R_1 \cap R_2$ by an invertible $\tilde{h}$ which satisfies $\zeta(t) \cdot \partial_x \tilde{h} = 0$ for $x \in \mathbb{R}^2$ and is analytic in $t$. As in the standard proof of Cartan’s Lemma, one can write

$$h = \prod_{k=1}^N e^{\psi_k}, \text{ with } \zeta(t) \cdot \partial_x \psi_k(x, t) = 0.$$ 

Hence, to approximate $h$ by an invertible matrix function it suffices to approximate the $\psi_k$’s. For $t = t_0$ the equation $\zeta(t) \cdot \partial_x h = 0$ reduces to $\partial_y \tilde{h} = 0$, and each entry $a$ in each $\psi_k$ is a function of $y_2$ alone. Hence for any $\beta > 0$ we can choose a polynomial $P(y_2)$ such that $|P(y_2) - a| < \beta$ and $|\frac{d}{dy_2}(P(y_2) - a)| < \beta$ on $R_1 \cap R_2$. Then, simply by continuity, there is a $\delta > 0$ such that $|\partial^2_x (P(\zeta(t) \cdot x) - a(x, t))| < 2\beta |a| < 1$, on $R_1 \cap R_2 \times \{ t : |t - t_0| < \delta, |t| \leq 1 \}$. Using these polynomials to approximate the entries in the $\psi_k$’s, we can construct $\tilde{h}$ such that $||\tilde{h}^{-1} h - I||_{1/2} < \beta'$ on $R_1 \cap R_2 \times \{ t : |t - t_0| < \delta' , |t| \leq 1 \}$ with $\tilde{h}$ invertible, analytic in $t$ and satisfying $\zeta(t) \cdot \partial_x \tilde{h} = 0$ globally in $x$. By taking $\beta$ sufficiently small we can make $\beta'$ small as we wish and still have $\delta' > 0$. In particular, for $\beta$ sufficiently small we have $\tilde{h}^{-1} h = \exp(\psi)$ with $||\psi||_{1/2}$ as small as we wish on $R_1 \cap R_2$.

Let $\mathcal{X} \in C^\infty(\mathbb{R})$ satisfy $\mathcal{X}(y_2) = 0$ on $R_1 \cap R_2^c$ and $\mathcal{X}(y_2) = 1$ on $R_2 \cap R_1^c$. Given $\alpha'', 0 < \alpha'' < \alpha'$, let $\rho \in C^\infty(\mathbb{R})$ be a cutoff function satisfying $\rho(y_1) = 1$ for $|y_1 - y_0| < \alpha''$ and $\rho(y_1) = 0$ for $|y_1 - y_0| > \alpha'$. Let $R_i^c, i = 1, 2$, be the narrower rectangles

$$R_i^c = \{ y : |y_1 - y_0| < \alpha'', a_i < y_2 < b_i \}.$$ 

Finally, we let $f_0$ be the solution to the integral equation

$$f_0 + \Pi(t) \rho(\zeta(t) \cdot \partial_x \mathcal{X}) \psi f_0 = I$$

(13)

Note by the proof of Lemma 1 that $f_0$ will exist and be invertible for $t \in \{ |t - t_0| < \delta' \} \cap \{ |t| \leq 1 \}$ for some $\delta' > 0$, provided that $\mathcal{X}$ is sufficiently small which is implied by $\beta''$ sufficiently small. Moreover,

$$\zeta(t) \cdot \partial_x (e^{\mathcal{X} \psi} f_0) = 0,$$

on $R_1^c \cap R_2^c$. We now have

$$C_2 e^{(\mathcal{X} - 1) \psi} f_0 = C_1 \tilde{h} e^{\mathcal{X} \psi} f_0 \quad \text{ on } R_1 \cap R_2.$$ 

(14)

Since $\tilde{h}$ and $f_0$ are globally defined, and $\mathcal{X} \psi$ and $(\mathcal{X} - 1) \psi$ can be extended to zero on $R_1 \cap R_2^c$ and $R_2 \cap R_1^c$ respectively, (14) shows that $C_2 e^{(\mathcal{X} - 1) \psi} f_0$ can be extended to a function $\tilde{C}_2$ on $R_1 \cup R_2$ which satisfies $\zeta(t) \cdot \partial_x \tilde{C}_2 = \zeta(t) \cdot A \tilde{C}_2$ on $R_1^c \cup R_2^c$.

To construct a solution on the strip $\{ y : |y_1 - y_0| < \alpha \}$ one applies this process repeatedly, beginning with $C_1 = I$ on the rectangle $\{ |y_1 - y_0| < 3\alpha/2, \ y_2 < -R \}$ and $C_2$ the solution from Lemma 1 on $\{ |y_1 - y_0| < 3\alpha/2, \ -R - \alpha/4 < y_2 < -R + 5\alpha/4 \}$. As one moves up the strip, adding squares to the domain of the solution, the $\alpha''$ for each square becomes the $\alpha'$ for the square above it. Thus one must choose a sequence of $\alpha''$’s beginning with $\alpha' = 3\alpha/2$ so that the last $\alpha''$ will be $\alpha$. Note that the last $C_2$ is just the identity matrix and the last rectangle $R_2^c$ is $\{ y : |y_1 - y_0| < \alpha, \ y_2 > R \}$.

The preceding gives us an invertible $C(x, t)$ satisfying (6) on $\{ y : |y_1 - y_0| < \alpha \} \times \{ |t - t_0| < \delta' \} \cap \{ |t| \leq 1 \}$ for some $\delta' > 0$. Moreover, $C$ is analytic in $t$.
on the intersection of its domain with \( \{ |t| < 1 \} \). The next step is the construction of solutions to (6) on all of \( \{ |x| < 2R \} \times \{ t : |t - t_0| < \delta', |t| \leq 1 \} \) from our solutions in strips. This is another application of what we have just done. One covers \( \{ x : |y|_1(x) | < R \} \) with strips of width \( \alpha = 2\alpha /3 \), and uses the solutions just constructed on the strips of width \( 3\alpha /2 \) with the same midpoints. Following the procedure precisely gives a solution of (6) on the domain \( \{ x : |y|_2 < 2R + 1 \} \). One begins with \( C_1 = I \) on the rectangle \( R_1 = \{ x : y_1 < -R, |y|_2 < 2R + 2 \} \). The smooth functions \( \rho \) and \( X \) are now functions of \( y_2 \) and \( y_1 \) respectively, and at every step we have \( \rho = 1 \) for \( |y|_2 < 2R + 1 \) and \( \rho = 0 \) for \( |y|_2 > 2R + 2 \) The support of \( \nabla X \) is now in the overlap of two adjacent strips.

At this point we have constructed a solution \( C(x, t) \) of (6) defined on \( \{ |y|_2 < 2R + 1 \} \times \{ |t - t_0| < \delta', |t| < 1 \} \) for each \( t_0 \) on \( \{ t_0 = 1 \} \), but to complete Step 1 we still need to show that these solutions satisfy (12). To prove (12) we need to show that the solutions \( C(x, t) \) themselves have the form \( C(x, t) = f(\zeta^{-1}(t) \cdot x, t) \) in suitable regions in \( (x, t) \) with \( f \) analytic in \( z \) on suitable regions in \( \mathbb{C} \). This requires a review of the construction via Cartan’s Lemma. Considering the terms in the version of (14) that gives the continuation from one strip to the next, note that by construction \( \hat{h}(x, t) = g(\zeta^{-1}(t) \cdot x) \) where \( g(z) \) is entire. The function \( f_0 \) is the solution of the integral equation

\[
f_0(x, t) + \frac{t}{\pi} \int_{\mathbb{R}^2} \rho(y_2(x'))(\zeta(t) \cdot \partial_x X(y_1(x'))))\psi(x', t)f_0(x', t)dx' = I. \tag{13'}
\]

Hence \( f_0(x, t) = F_0(\zeta^{-1}(t) \cdot x, t) \) where

\[
F_0(z, t) = I - \frac{t}{\pi} \int_{\mathbb{R}^2} \rho(y_2(x'))(\zeta(t) \cdot \partial_x X(y_1(x')))\psi(x', t)f_0(x', t)dx'.
\]

Let

\[ B(t) = \{ z \in \mathbb{C} : z = \zeta^{-1}(t) \cdot x', |y_1(x')| \leq 3R/2, |y_2(x')| \leq 2R + 2 \}, \]

where \( (y_1, y_2) \) is defined as \( (y_1, y_2) \) with \( \theta_0 \) replaced by the argument of \( t \). Hence \( B(t) \) does not depend on \( t_0 \). Since the support of \( \rho \nabla X \) is contained in \( \{ |y|_2 \leq 2R + 2, |y|_1 \leq R + 1 \} \), we have

\[ \{ z \in \mathbb{C} : z = \zeta^{-1}(t) \cdot x', x' \in \text{supp}(\rho \nabla X) \} \subset B(t). \]

Thus \( F_0(z, t) \) is analytic in \( (z, t) \) on \( \{ (z, t) : z \in B(t)^c, |t - t_0| < \delta, |t| < 1 \} \). We think of \( B(t) \) as the “bad set” where \( F_0(z, t) \) may fail to be analytic in \( z \). As \( r \to 1 \), \( B(re^{i\theta}) \) converges to the line segment \( L(\theta) = \{ e^{i\theta} : -2R - 2 < s < 2R + 2 \} \) (see the definition of \( \zeta^{-1}(t) \) before (12)). Since \( \psi \) and \( f_0 \) converge as \( r \to 1 \), it follows from Privalov’s Theorem on Cauchy integrals that \( F_0(z, re^{i\theta}) \) converges to an analytic function \( F_0(z, e^{i\theta}) \) on \( L(\theta)^c \) with continuous extensions (not necessarily equal) to \( L(\theta) \) from either side.

We will use the disjoint subsets of \( (B(t))^c \)

\[
S_+(t) = \{ z \in (B(t))^c : \text{Im}\{e^{-i\theta}z\} > 0, |\text{Re}\{e^{-i\theta}z\}| \leq 2R + 1 \} \text{ and } S_-(t) = \{ z \in (B(t))^c : \text{Im}\{e^{-i\theta}z\} < 0, |\text{Re}\{e^{-i\theta}z\}| \leq 2R + 1 \}.
\]

Letting \( \tilde{C}_k \), \( k = 1, \ldots, M \), and \( \tilde{C}_0 = C_1 = I \) be the sequence of solutions from Cartan’s Lemma, at each step we have

\[
\tilde{C}_k(x, t) = \tilde{C}_{k-1}\tilde{h}_k f_0, k \text{ for } y_h(x) < -3R/2. \tag{15}
\]
Note that the factor \( e^{X \psi} \) does not appear in (15) because \( X = 0 \) on this set. Therefore (15) implies that for \( \zeta(t) \cdot x \in S_+(t) \) we have \( C(x,t) = f_+(\zeta(t) \cdot x, t) \), where \( f_+(z,t) \) like \( F_0 \) is analytic in \( z \) on \( S_+(t) \). Likewise, since \( C_M \) is the identity matrix on \( \{ x : y_l > R \} \), we have \( C_M = f_0,M \) for \( y_l > 3R/2 \). Hence, for \( \zeta(t) \cdot x \in S_+(t) \) we have \( C(x,t) = f_+ (\zeta(t) \cdot x, t) \), where \( f_+(z,t) \) like \( F_0 \) in analytic in \( z \) on \( S_+(t) \). These representations of \( C(x,t) \) on \( S_+(t) \) are what we need to proceed with the proof of (12).

To prove (12) suppose that \( C_1 \) and \( C_2 \) are two of the local solutions in \( t \) that we have just constructed. Then \( h = C_1^{-1} C_2 \) is a solution of \( \zeta(t) \cdot \partial_t h = 0 \) for \( x \) in the intersection of two closed strips of width \( 4R + 2 \) and \( t \) in the intersection of two small disks with centers on \( |t| = 1 \). For \( \zeta(t) \cdot x \in S_+(t) \) we have \( h(x,t) = g_+ (\zeta(t) \cdot x, t) \), where \( g_+ \) and \( g_- \) have the properties of \( f_+ \) and \( f_- \) in the preceding paragraph. Moreover, \( \zeta(t) \cdot \partial_t h = 0 \) on the intersection of the strips. For \( \delta \) sufficiently small this intersection contains the square \( \{ |y_1(x)| \leq 2R, |y_2(x)| \leq 2R \} \). Hence, \( h(x,t) = \tilde{g} (\zeta(t) \cdot x, t) \), where \( \tilde{g}(z,t) \) is analytic on the set

\[
\tilde{S}(t) = \{ z \in \mathbb{C} : z = \zeta(t) \cdot x, \ |y_1(x)| \leq 2R, \ |y_2(x)| \leq 2R \}.
\]

From the definitions of \( S_+(t) \) and \( S_-(t) \), one sees that, for \( |t| < 1 \), \( \tilde{S}(t) \) has nonempty open intersections with both \( S_+(t) \) and \( S_-(t) \). Hence \( g(z,t) \) has an analytic continuation in \( z \) (given by \( g_+(z(t),t) \) to the left and \( g_+(z(t),t) \) to the right) to all of \( S_+(t) \cup S_-(t) \cup \tilde{S}(t) \). Since \( \{ |z| \leq 2R \} \subset S_+(t) \cup S_-(t) \cup \tilde{S}(t) \) for \( |t| \) close to 1, we have the representation of \( C_1(t)^{-1} C_2(t) \) in (12) for \( |t| < 1 \). Finally, since \( h(x,t) \) has a continuous extension to \( |t| = 1 \) as a function with values in \( C^{1/2} (|x| \leq 2R+1) \), it follows that the limiting values of \( g_+(z, r e^{i \theta}) \) on \( \mathbb{L}(\theta) \) as \( r \to 1 \) must agree, so that

\[
g(z, e^{i \theta}) = \begin{cases} 
g_+(z, e^{i \theta}) & \text{on } S_+(e^{i \theta}) \\
g_-(z, e^{i \theta}) & \text{on } S_-(e^{i \theta}) \end{cases}
\]
defines an analytic function on \( |z| \leq 2R \). This completes the derivation of the property (12).

The construction of solutions for \( |t_0| < 1 \) is simpler, because we can work on the disks \( D(t_0) = \{ t : |t - t_0| \leq \frac{1}{2} |1 - |t_0|| \} \). For \( t \) in this disk \( \zeta(t) \cdot \partial_t \) is nondegenerate and equivalent to \( \partial_\nu \) by a linear change of variables for \( t \). Thus, one can use Runge’s Theorem to construct \( \tilde{h} \) approximating \( h = C_1^{-1} C_2 \) on the intersection in \( x \) of the domains of \( C_1 \) and \( C_2 \) uniformly in \( t \). Thus, we are in the setting for Cartan’s Lemma, and can apply it to construct solutions of (6) in large disks \( \{ |x| < R(t_0) \} \) where \( R(t_0) \) is chosen so that

\[
\{ |z| \leq 2R \} \subset \{ z = \zeta(t) \cdot x, \ |x| < R(t_0) \}
\]

for all \( t \in D(t_0) \).

Define

\[
P(x, t) = C(x, t) = [\Pi(t) A(\cdot, \cdot) C(\cdot, t) (\cdot, t)](x). \tag{16}
\]

Then, since \( \zeta(t) \cdot \partial_t P = 0 \) on \( |x| < R(t_0) \), we have \( P = \tilde{g} (\zeta(t) \cdot x, t) \) with \( \tilde{g}(z,t) \) analytic on \( \{ |z| \leq 2R \} \times \mathbb{D}(t_0) \). Also, since \( A \) is supported in \( \{ |x| < R \} \),

\[
[\Pi(t) A(\zeta(t) C(\cdot, t)) (x)] = g(\zeta(t) \cdot x, t) \text{ where } g(z, t) \text{ is analytic in } z \text{ on } B_1(t) \text{ with
}
B_1(t) = \{ z \in \mathbb{C} : z = \zeta(t) \cdot x, \ |x| \leq R \} \subset B(t).
\]

Thus we have \( C(x, t) = f(g(\zeta(t) \cdot x, t)) \) for \( \zeta(t) \cdot x \in \{ |z| \leq 2R \} \setminus B_1(t) \). This domain is large enough that property (12) will hold on the intersections of the
domains of the solutions constructed here, and their intersections with the domains of the solutions constructed for $|t_0| = 1$.

**Step 2.** Now for each $t_0$ with $|t_0| \leq 1$, we have an invertible solution $C(x, t)$ of (6) defined on $\{ |z| < 2R + 1 \} \times \{ |t - t_0| < \delta(t_0), \ |t| \leq 1 \}$ for some $\delta(t_0) > 0$. All these solutions are analytic in $t$ on the intersections of their domains with $\{ |t| < 1 \}$, and they have the property that on the intersection of the domains of any pair of solutions, $C_i$ and $C_j$, $C_i^{-1} C_j(x, t) = h_{ij}(\zeta(t) \cdot x, t)$, where $h_{ij}(z, t)$ is analytic in $z$ on $|z| \leq 2R$. Now we can consider the $h_{ij}$’s as defining a holomorphic fiber bundle — with fiber the holomorphic mappings of $|z| \leq 2R$ into $GL(n, \mathbb{C})$ — over $\{ |t| < 1 \}$, and it has a continuous extension to $|t| = 1$. The standard form of Cartan’s Lemma — see [9] or [10] — says that the bundle is trivial, i.e. for any cover $\{ U_{ij} \}_{j=1}^M$ of $\{ |t| \leq 1 \}$ by simply connected open sets with functions $h_{ij}$ on $U_i \cap U_j$, satisfying $h_{ij} h_{jk} h_{ki} = I$ on $U_i \cap U_j \cap U_k$, there are are functions $h_i$ on $U_i$ and $h_j$ on $U_j$ such that $h_{ij} = h_i h_j^{-1}$.

We choose a cover of $|t| \leq 1$ by disks $D_j = \{ |t - t_0^j| \leq \delta_j \}$, $j = 1, \ldots, M$, with the property that $(\cup_{j=1}^k D_j) \cap D_{k+1}$ is simply connected for each $k$. Then, applying the preceding result on the sequence of covers

$$C_k = \{ \cup_{j=1}^k D_j, D_{k+1}, \ldots, D_M \},$$

as in Step 1, we can construct $C(x, t)$, an invertible solution of (6) on all of $\{ |z| \leq 2R \} \cap \{ |t| < 1 \}$. To see that this contraction also succeeds when $|t| \leq 1$, we need to recall part of the argument in Step 1. To get the factorization of $h_{ij}$ one first writes it as a product of exponentials and then appeals to Runge’s Theorem to approximate the exponents by functions which are polynomial in $t$ and holomorphic in $z$ on $|z| \leq 2R$. This is not possible when $|t| = 1$, so we first choose $\alpha < 1$ so that $|h_{ij}(z, \alpha t)^{-1} h_{ij}(z, t) - I| < \beta / 2$ on the intersection and then apply Runge’s Theorem to the exponents in the representation of $h_{ij}(z, \alpha t)$ to construct $h_{ij}$. Also the support of $\nabla \mathcal{X}$ cannot be contained in $\{ |t| < 1 \}$. Since only $\| \partial_p \mathcal{X} \psi \|_\infty$ needs to be small when we solve for $f_0$, it suffices to extend $\nabla \mathcal{X}$ by 0 for $|t| \geq 1$ in the equation for $f_0$. Thus the factorization $h_{ij} = h_i h_j^{-1}$ with

$$h_i = e^{X_{ij} \psi_j} f_{0,ij}$$

and $h_j = e^{(X_{ij} - 1) \psi_j} f_{0,ij}$

as in (14) extends continuously to $|t| = 1$. This completes the proof of Theorem 5.

**Remark.** Assume that $|t| \geq 1/2$. Let $C(x, t)$ be the solution of (6) constructed in Theorem 5, and define $P(x, t)$ by (16). Since $C$ is constructed by multiplying the local (in $t$) solutions from Step 1 by functions $h_j(\zeta(t) \cdot x, t)$, where $h_j(z, t)$ is analytic for $|z| \leq 2R$, we still have $C(x, t) = f_{\pm}(\zeta(t) \cdot x, t)$ for $\zeta(t) \cdot x \in S_\pm(t) \cap \{ |z| \leq 2R \}$ as in Step 1. Note that $B(t)$, and hence $S_\pm(t)$, are defined for $t$ in a small neighborhood of any $t_0$ with $|t_0| > 0$. Since $\zeta(t) \cdot \partial_t P = 0$ on the domain of $C$, we can conclude that, for $\zeta(t) \cdot x \in \hat{S}(t)$, $P(x, t) = \hat{g}(\zeta(t) \cdot x, t)$, where $g(z, t)$ is analytic in $z$ on $\hat{S}(t)$. For $|t| < 1$ the intersections of $S(t)$ with $S_+(t)$ and $S_-(t)$ are nonempty open sets. Hence, by exactly the argument used to verify (12) in Step 1, we conclude that $P(x, t) = \hat{g}(\zeta(t) \cdot x, t)$, where $\hat{g}(z, t)$ is analytic in $z$ on $|z| \leq 2R$ for all $t$ with $1/2 \leq |t| \leq 1$. This property of $P$ is used in [3].

**An Example**

It is quite easy to find matrices $B$ such that equation (3) above has no solutions which tend to the identity as $|z| \to \infty$. One example is the following.
Choose $\phi_i \in C^\infty(\mathbb{R}^2)$, $i = 1, 2$, such that $\phi_1 = 1$ for $|z + 3| > 2$ and $\phi_1 = 0$ for $|z + 3| < 1$; $\phi_2$ has the same properties with 3 replaced by -3. Then, noting that $\phi_2 = 1$ on the support of $\partial_\nu \phi_1$ and vice versa, one sees that

$$u = \left( \frac{\phi_1}{z+3}, \frac{\phi_2}{z-3} \right)$$

satisfies $0 = (\partial_z + B)u$, where

$$B = \begin{pmatrix} 0 & -\left(\partial_\nu \phi_1\right) \frac{z+3}{z-3} \\ -\left(\partial_\nu \phi_2\right) \frac{z+3}{z-3} & 0 \end{pmatrix}.$$

Thus for this matrix $B$ the function $u$ is a nontrivial solution to $(\partial_z + B)u = 0$ with compact support. As was observed in [2], if there were an invertible solution $C$ to (1') tending to the identity at infinity, $C^{-1}u$ would be a analytic function on $\mathbb{R}^2$ which vanished for $|z|$ large. Thus $u$ would be identically zero, contradicting the above. Finally one notes that for any solution of (1') $\det C = f(z) \exp(\Pi \text{tr}(B))$ where $f$ is entire. Hence any solution of (1') which tends to the identity as $|z| \to \infty$ must be invertible for all $z$.

References


