Gronwall's Inequality

We suppose that we have two solutions, $x_1(t)$ and $x_2(t)$, to the differential equation $x' = f(x,t)$, and we want an estimate of $|x_1(t) - x_2(t)|$ in terms of $|x_1(t_0) - x_2(t_0)|$. Think of this as measuring how much the two solutions can move away from each other in terms of how far apart they are when $t = t_0$. We look at the equation on a rectangle $R = \{a < t < b,\ c < x < d\}$, and use the number M,

$$M = \max_{(t,x)\in R} |\frac{\partial f}{\partial x}(t,x)|$$

in the estimate. The estimate is

$$|x_1(t) - x_2(t)| \le |x_1(t_0) - x_2(t_0)| e^{M|t-t_0|}. \tag{0}$$

This holds as long as the points $(t, x_1(t))$ and $(t, x_2(t))$ stay in R. Notice that when $x_1(t_0) = x_2(t_0)$ the estimate says that $x_1(t) - x_2(t) \equiv 0$. That's the uniqueness theorem. In the examples where uniqueness fails, like $x' = x^{2/3}$, you will always find $\partial f/\partial x$ is unbounded, and the estimate doesn't say anything when $M = \infty$.

Anyway...here is how you derive the estimate. I will derive it when $t > t_0$. You have to change some signs when $t < t_0$, but the derivation is really the same.

Step 1: Integrate the equations $x_1' = f(t, x_1(t))$ and $x_2' = f(t, x_2(t))$ from t_0 to t. That gives

$$x_1(t) = x_1(t_0) + \int_{t_0}^{t} f(s, x_1(s))ds, \text{ and } x_2(t) = x_2(t_0) + \int_{t_0}^{t} f(s, x_2(s))ds.$$

Now subtract the second equation from the first to get

$$x_1(t) - x_2(t) = x_1(t_0) - x_2(t_0) + \int_{t_0}^{t} [f(s, x_1(s)) - f(s, x_2(s))]ds. \tag{1}$$

Step 2: Take absolute values of both sides of (1), and estimate. This step uses some very basic facts about absolute values that you may have not seen before. First, there is the "triangle inequality": $|A + B| \le |A| + |B|$, and then there is the inequality

$$|\int_{t_0}^{t} h(s)ds| \le \int_{t_0}^{t} |h(s)|ds.$$

Using both of those on (1), you get

$$|x_1(t) - x_2(t)| \le |x_1(t_0) - x_2(t_0)| + \int_{t_0}^{t} |f(s, x_1(s)) - f(s, x_2(s))|ds. \tag{2}$$

Step 3: Use the Mean Value Theorem in the variable x. The mean value theorem says

$$f(s, x_1(s)) - f(s, x_2(s)) = (x_1(s) - x_2(s))\frac{\partial f}{\partial x}(s, x_*(s)), \tag{3}$$

where $x_*(s)$ is a point between $x_1(s)$ and $x_2(s)$ – which means that $(s, x_*(s))$ is in R. Taking absolute values of both sides of (3) and replacing $|\partial f/\partial x(s, x_*(s))|$ by M, you get

$$|f(s, x_1(s)) - f(s, x_2(s))| \leq |x_1(s) - x_2(s)|M.$$

Substituting that back into (2), you get

$$|x_1(t) - x_2(t)| \leq |x_1(t_0) - x_2(t_0)| + M\int_{t_0}^{t} |x_1(s)) - x_2(s))|ds, \tag{4}$$

Step 4: Now we come to Gronwall's great idea. Define

$$F(t) = \int_{t_0}^{t} |x_1(s)) - x_2(s))|ds \text{ and observe that } F'(t) = |x_1(t) - x_2(t)|.$$

That means that you can rewrite (4) as

$$F'(t) \leq |x_1(t_0) - x_2(t_0)| + MF(t),$$

and multiplying that by an integrating factor

$$e^{-M(t-t_0)}F'(t) - Me^{-M(t-t_0)}F(t) \leq e^{-M(t-t_0)}|x_1(t_0) - x_2(t_0)|$$

which means

$$\frac{d}{dt}(e^{-M(t-t_0)}F) \leq e^{-M(t-t_0)}|x_1(t_0) - x_2(t_0)|. \tag{5}$$

Since $F(t_0) = 0$, you can integrate both sides of (5) from t_0 to t to get

$$e^{-M(t-t_0)}F(t) \leq \frac{1}{M}(1 - e^{-M(t-t_0)})|x_1(t_0) - x_2(t_0)|$$

or

$$F(t) \leq \frac{1}{M}(e^{M(t-t_0)} - 1)|x_1(t_0) - x_2(t_0)| \tag{6}$$

Finally, substituting (6) into (4), you have exactly

$$|x_1(t) - x_2(t)| \leq |x_1(t_0) - x_2(t_0)|e^{M(t-t_0)}$$

which is the estimate (0) in the case $t > t_0$.