Inverse Scattering for Gratings and Wave Guides

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Abstract: We consider the problem of unique identification of dielectric coefficients for gratings and sound speeds for wave guides from scattering data. We prove that the “propagating modes” given for all frequencies uniquely determine these coefficients. The gratings may contain conductors as well as dielectrics and the boundaries of the conductors are also determined by the propagating modes.

§0. Introduction

Consider Maxwell’s equations for time-harmonic electric and magnetic fields, exp(−iωt)E(x1, x2, x3) and exp(−iωt)H(x1, x2, x3), in the absence of currents and charges

\[ \nabla \times E - i\omega \mu_0 H = 0 \]
\[ \nabla \times H + i\omega \epsilon(x) E = 0. \]

In this paper we study the inverse problem of determining the electric permittivity, ϵ, and hence the dielectric coefficient, ϵ/ϵ_0, from scattering data for these equations. The fundamental assumptions are that ϵ is independent of x_3, 2π-periodic in x_1 and constant (= ϵ_0) for |x_2| > T. These conditions are designed to model a dielectric “grating” extending throughout the region |x_2| < T (c.f. [P], [BDC]). We also allow for conducting bodies embedded in the dielectric as long as they satisfy conditions analogous to our conditions on ϵ: their boundaries should be invariant with respect to all translations in x_3 and translations by 2π in x_1, and they should be contained in |x_2| < T.

To define data sets for this inverse problem it is customary to consider the scattering problem for fields with either the transverse electric (TE) or transverse magnetic (TM) polarizations, respectively E(x) = (0, 0, u(x_1, x_2)) and H(x) = (0, 0, v(x_1, x_2)). These polarizations reduce Maxwell’s equations, respectively, to

\[ \frac{1}{\epsilon(x)} \Delta u + \omega^2 \mu_0 u = 0 \]  (TE)

and

\[ \nabla \cdot \frac{1}{\epsilon(x)} \nabla v + \omega^2 \mu_0 v = 0. \]  (TM)

In the case of embedded conductors we consider the TE polarization in the exterior of the conductors with the Dirichlet condition, u = 0, on the boundary, since this corresponds to E = 0 and H · \hat{n} = 0 on the surfaces of the conductors (here \hat{n} denotes the unit outer normal). For our purposes it is convenient to write both (TE) and (TM) as

\[ Lu - k^2 u = 0, \]  (I.1)
where \( L = -\Delta \) for \(|x_2| > T\) and \( k^2 = \omega^2 \mu_0 \epsilon_0 \) with \( k > 0 \).

We will also present the analogous inverse problem for acoustic wave guides. This requires only small modifications of the arguments for gratings. The wave guides that we consider are simply slabs, \( \{0 < x_1 < B\} \), in which the sound speed \( c \) is a function of \((x_1, x_2)\). We assume that \( c(x_1, x_2) = c_0(x_1) \) for \(|x_2| > T\), and impose Dirichlet condition on \( x_1 = 0 \), and the Neumann condition, \( \partial_{x_1} u = 0 \), on \( x_1 = B \). These boundary conditions correspond to an acoustically soft reflecting surface at \( x_1 = 0 \) and an acoustically hard reflecting surface at \( x_1 = B \), modelling underwater sound propagation with \( x_1 \) as depth (c.f. [BGWX]). We will show that scattering data from propagating modes for the operator \( L = -c^2(x)\Delta \) with these boundary conditions determine \( c(x) \).

In both these settings we will apply recent results on inverse coefficient problems for hyperbolic equations (Belishev [B], Kachalov-Kurylev-Lassas [KKL] and Eskin [E1],[E2]). In those papers the data for the inverse problem is the Dirichlet-to-Neumann map. Hence the objective here will be to show that the scattering data determine the Dirichlet-to-Neumann map on a line \( x_2 = T \).

The “scattering data from propagating modes” that we use in these inverse problems will be defined in \( \S 1 \). Our data are non-evanescent components of the scattered waves. Only such components can be measured easily in practical situations. To the best of our knowledge the present paper is the first to show that a variable dielectric is uniquely determined by these data.

Inverse scattering problems for dielectric gratings have been studied previously in [BDC], [BF], [K1], [HK] and [EY]. These articles consider primarily the inverse problem of finding the boundaries of conductors embedded in a dielectric of constant permittivity from scattering data. In [K2] Kirsch studies scattering from a dielectric with varying incident direction rather varying frequency.

Inverse coefficient problems for wave guides were studied in [BGMX], [GMX], [H], [X] and [DM]. These papers give methods for recovering the sound speed. We only consider the uniqueness problem and prove that the sound speed is uniquely determined by the propagating modes. Our approach was influenced by the work of S.Dediu and J. McLaughlin, [DM], which also uses propagating modes.

\( \S 1. \) Statement of results

Our results for gratings hold under mild conditions on the operator on \( L \) in (I.1). We will assume that \( L \) is a second order elliptic operator on \( D \subset \mathbb{R}^2 \) which is symmetric in the inner product

\[ (f, g) = \int_D f(x)g(x)a(x)dx. \]

The weight \( a(x) \) is continuous and strictly positive on \( \overline{D} \). The coefficients of \( L + \Delta \) are supported in \(|x_2| < T\), and \( L \) commutes with translation by \( 2\pi \) in \( x_1 \), i.e. \( L = -\Delta \) for \(|x_2| \geq T\), and the coefficients of \( L \) are \( 2\pi \)-periodic in \( x_1 \).

Likewise \( a(x) - 1 \) is supported in \(|x_2| < T\) and \( a(x_1 + 2\pi, x_2) = a(x_1, x_2) \). We will also assume that the region \( D \) is invariant under translation by \( 2\pi \) in \( x_1 \), and boundary \( D \) is smooth. We consider two types of gratings:

Case 1: \( D \) is connected and contains \( \{|x_2| > T\} \). In other words, while there may be some holes in \( D \), they do not disconnect \( D \), and they are contained in \(|x_2| < T\).
Case 2: $D$ is connected and we have the inclusions 
$$ \{x_2 > T\} \subset D \subset \{x_2 > -R\} $$
for some $R > 0$.

Let $0 \leq \alpha < 1$. The domain of $L$ will be $H_0^2(D) \cap H_1^0(D)$. By $H_0^k(D)$ (resp. $H_{a,0}(D)$) we mean functions satisfying
$$ u(x_1 + 2\pi, x_2) = e^{2\pi i a} u(x_1, x_2) $$
for some $\alpha > 0$, for some $\phi \in C_0^\infty(\mathbb{R})$. Note that
$$ \lim_{\text{Im} k \to -\infty} (L - k^2)u = 0 $$
for all $\phi \in C_0^\infty(\mathbb{R})$.

Let this domain for $L$ corresponds to the Dirichlet boundary condition on $\partial D$.

For wave guides we simply take $L = -(x_1^2 - k^2)v(x_1, x_2)$ on $D = \{x_1 < x_2 < \} \{x_1 < B\}$, with Dirichlet and Neumann boundary conditions on $x_1 = 0$ and $x_1 = B$, respectively. As indicated above, $c(x) = c_0(x_1)$ when $|x_2| > T$.

For both gratings and wave guides scattering data at fixed energy $k^2$ are obtained from the “propagating modes”. Let $l = (l_1, l_2)$, $l_1 = n + \alpha$, $l_2 = -\sqrt{k^2 - (n + \alpha)^2}$, $n \in \mathbb{Z}$. In the case of gratings we assume that the scattered wave $v_+ = v_+(x, n, k)$ satisfies
$$ (L - k^2)v_+ = -(L - k^2)e^{i\xi x} $$
and the radiation condition for large $|x_2|$: 
$$ v_+(x, n, k) = \sum_{m \in \mathbb{Z}} e^{i[(m+\alpha)x_1 + x_2\sqrt{k^2 - (m+\alpha)^2}]} a_m(n, k), $$
provided that $k$ does not belong to the set of “thresholds”, $\{k : k^2 = (p+\alpha)^2, p \in \mathbb{Z}\}$. This means that $v_+$ is the “outgoing” solution to
$$ (L - k^2)v_+ = -(L - k^2)e^{i\xi x} $$
obtained as the limit as Im$\{k\} \to 0^+$ (see below), and we can write
$$ v_+(x, n, k) = \sum_{(n+\alpha)^2 < k^2} e^{i[(m+\alpha)x_1 + x_2\sqrt{k^2 - (m+\alpha)^2}]} a_m(n, k) + O(e^{-\delta x_2}) $$
for some $\delta > 0$. We call \{a_m(n, k), n, m \in \mathbb{Z} : (m + \alpha)^2 < k^2, (n + \alpha)^2 < k^2\} the scattering data at energy $k^2$ from “propagating modes”. Thus our scattering data are non-evanescent components of the scattered wave and from the practical viewpoint, only such non-evanescent components can be easily measured.

For wave guides, since we are taking $L_0 = -c_0(x_1^2)\Delta$ as the unperturbed operator, the scattered wave $v_+ + v_+(x, n, k)$ is obtained by solving $(L - k^2)v = 0$ with $L = \Phi(x, n, k) + v_+$, where $\Phi$ is a generalized eigenfunction for $L_0$, i.e. $\Phi(x, n, k) = \exp(-ix_2\sqrt{\mu_n(k)})\phi_n(x_1, k)$, where
$$ \phi'' + k^2 c_0^{-2} \phi_n = \mu_n(k) \phi_n, \phi_n(0) = 0, \phi'_n(\pi) = 0, \mu_n(k) > 0. $$
For $x_2 > T$ the scattered wave $v_+$ has the form
$$ v_+(x, n, k) = \sum_{m \in \mathbb{N}} b_m(n, k) e^{im\xi x_2\sqrt{\mu_n(k)}} \phi_m(x_1, k) $$
D and $\mu_{n}(k)$ determine the Dirichlet-to-Neumann map on a suitable line

Note that in these definitions the functions $\sqrt{k^2 - (m + \alpha)^2}$ and $\sqrt{\mu_{n}(k)}$ will be chosen so that they extend into $\text{Im}\{k\} > 0$ with positive imaginary parts. Letting $z(k)$ stand for either $k^2 - (m + \alpha)^2$ or $\mu_{n}(k)$, this choice amounts to choosing $\sqrt{z(k)} > 0$ when $z(k) > 0$ and $k > 0$, $\sqrt{z(k)} < 0$ when $z(k) > 0$ and $k < 0$, and $\sqrt{z(k)} = i\sqrt{|z(k)|}$ when $z(k) < 0$. We will follow these conventions in the rest of the paper.

With the preceding definitions we have:

**Theorem 1:** The scattering data from propagating modes given for all $k$ determine $D$ and $\epsilon(x)$ for gratings with either the (TE) or (TM) polarizations, and $\epsilon(x)$ for wave guides.

The proof of theorem will proceed as follows. We will consider “generalized distorted plane waves”

$$u_{+}(x, n, k) = \exp(i(n + \alpha)x_1 - ix_2\sqrt{k^2 - (n + \alpha)^2}) + v_{+}(x, n, k)$$

and

$$u_{+}(x, n, k) = \exp(-ix_2\sqrt{\mu_{n}(k)})\phi_{n}(x, k) + v_{+}(x, n, k),$$

which are defined without the restrictions $k^2 > (n + \alpha)^2$ and $\mu_{n}(k) > 0$. These generalized distorted plane waves exist for $k \in \mathbb{R}\setminus S$, where $S$ is a discrete set. Note that when $k^2 < (n + \alpha)^2$ or $\mu_{n}(k) < 0$ these generalized distorted plane waves grow exponentially as $x_2 \to \infty$. In §2 and §3 we show that the set of generalized distorted plane waves, given for a fixed $k$ and all $n$, uniquely determines the Dirichlet-to-Neumann map on a suitable line $x_2 = T$ for all choices of $k$ outside a discrete set. We also show that, making use of the analytic continuation to $\text{Im}\{k\} > 0$ of the $v_{+}(x, n, k)$’s, these generalized distorted plane waves are determined by the scattering data from propagating modes. Thus, under the hypotheses of Theorem 1, the Dirichlet-to-Neumann map is known on $x_2 = T$ for all $k \in \mathbb{R}$ outside a discrete set. Since this is equivalent to knowing the *hyperbolic* Dirichlet-to-Neumann map for the wave equations $u_{tt} = Lu$ on $x_2 = T$, the proof of Theorem 1 will be reduced to the results on hyperbolic inverse coefficient problems cited above. Since analytic continuation plays a big role here, there are many variations on the set of $k$ for which the propagating modes are known which lead to the same results. For instance, one only needs to know the coefficient $a_{m}(n, k)$ in (2) on an open interval in $\{k > 0 : k^2 > \text{max}\{(n + \alpha)^2, (m + \alpha)^2\}\}$.

§2. Determination of the Dirichlet-to-Neumann Map for Gratings

In this section we will show that the scattering data from propagating modes determine the Dirichlet-to-Neumann map on a line $x_2 = T$ for the case of gratings. To do this we will first show that the traces of an appropriate family of distorted plane waves on $x_2 = T$ are dense in $L^2(0 < x_1 < 2\pi)$.
To begin we need the incoming and outgoing fundamental solutions for $-\Delta - k^2$ on $\mathbb{R}^2$ in a form compatible with (1). Using Fourier series in $x_1$ to reduce this to an ODE in $x_2$, one computes that for $\text{Im}\{k\} > 0$

$$[(\Delta - k^2)^{-1}f](x) =$$

$$\sum_{m=-\infty}^{\infty} \frac{1}{2\pi} \int_{-\infty}^{\infty} \left( \int_{0}^{2\pi} e^{i\lambda(k)(x_1-y_1)+\lambda_m(k)x_2-y_2} \frac{f(y)}{2i\lambda_m(k)} dy_1 \right) dy_2,$$

where

$$\lambda_m(k) = \sqrt{k^2 - (m+\alpha)^2}$$

with the branch chosen ≈ $k$ near infinity and the cut on $(|m+\alpha|, |m+\alpha|)$. Note that $\text{Im}\{\lambda_m(k)\} > 0$ for $\text{Im}\{k\} > 0$, and hence $(-\Delta - k^2)^{-1}$ maps $H^0_0(\mathbb{R}^2)$ into $H^2_0(\mathbb{R}^2)$. The continuous extension of $\lambda_m(k)$ from $\text{Im}\{k\} > 0$ to the real axis is given by

$$\lambda_m(k) = k\sqrt{1 - (m+\alpha)^2/k^2}$$

when $(m+\alpha)^2 < k^2$ and

$$\lambda_m(k) = i\sqrt{(m+\alpha)^2 - k^2}$$

when $(m+\alpha)^2 > k^2$. The corresponding extension of $(-\Delta - k^2)^{-1}$ to $\mathbb{R}\\{\pm(m+\alpha), m \in \mathbb{Z}\}$ gives the outgoing fundamental solution, $G_+(k)$. For the incoming fundamental solution we take

$$\lambda_m(k) = -\sqrt{k^2 - (m+\alpha)^2},$$

i.e. the branch chosen $\sim -k$ near infinity. Substituting this in the formula for $(-\Delta - k^2)^{-1}f$ to get $(-\Delta - k^2)^{-1}$ in $\text{Im}\{k\} < 0$, and define the incoming fundamental solution, $G_-(k)$, by continuous extension from $\text{Im}\{k\} < 0$ to the real axis. Hence, by construction $G_+(k)$ extends analytically to $(-\Delta - k^2)^{-1}$ in $\text{Im}\{k\} > 0$, and $G_-(k)$ extends analytically to $(-\Delta - k^2)^{-1}$ in $\text{Im}\{k\} < 0$.

Now we turn to the construction of generalized distorted plane waves for $L$. Choose $\psi \in C^\infty(\mathbb{R})$ such that $\psi \equiv 1$ on a neighborhood of $\{|x_2| \geq T\}$, and the support of $\psi(x_2)$ is contained in the set where $L = -\Delta$ and $a = 1$. An (outgoing) generalized distorted plane wave for $L$ is a solution of $(L - k^2)u = 0$ in $D$ of the form $u_+ = \psi(x_2)e^{i\ell \cdot x} + v_+$, where $\ell_1 \equiv \alpha \mod 1$ with $0 \leq \alpha < 1$, $\ell_2 = -\sqrt{k^2 - (m+\alpha)^2}$, and $v_+$ is defined by limiting absorption (see Appendix I), i.e.

$$v_+ = -\lim_{\epsilon \to 0^+} (L - (k + i\epsilon)^2)^{-1}(L - k^2)(\psi(x_2)e^{i\ell \cdot x}).$$

These are generalized distorted plane waves in the sense used in §1, since the second component of $l$ is not necessarily real. Ordinarily, outgoing distorted plane waves are defined as solutions of the form $u_+ = \exp(i\ell \cdot x) + v_+$ where $v_+$ is outgoing. However, we have

$$u_+ = \psi(x_2)e^{i\ell \cdot x} + v_+ = e^{i\ell \cdot x} + [\psi(x_2) - 1]e^{i\ell \cdot x} - \lim_{\epsilon \to 0^+} (L - (k + i\epsilon)^2)^{-1}(L - k^2)(\psi(x_2)e^{i\ell \cdot x}).$$
Since \((\psi(x_2) - 1) \exp(i\ell \cdot x)\) is outgoing, the term in brackets is outgoing.

The limit defining \(v_+\) will exist unless

i) \(k\) is one of the “thresholds”, \(k^2 = (n + \alpha)^2\), where \(G_{\pm}(k)\) are undefined, or

ii) there is a solution to the homogeneous equation \((L - k^2)u = 0\) in \(D\) which is square-integrable in \(D \cap \{0 < x_1 < 2\pi\}\).

We denote the set of exceptional \(k\)'s defined by i) and ii) as \(S\).

Since \(l_1 = n + \alpha\) for a unique \(n \in \mathbb{Z}\), we use \(n\) and \(k\) to parametrize the generalized distorted plane waves, \(u = u(x, n, k)\). With these definitions we have outgoing distorted plane waves for all \((n, k) \in \mathbb{Z} \times \mathbb{R} \setminus S\). The analytic properties of \(G_+(k)\) discussed above carry over to the \(u_+(x, n, k)'s:\) they have analytic continuations to \(\text{Im}\{k\} > 0\) which extend continuously back to \(\mathbb{R} \setminus S\). This leads directly to the following conclusion which we state as a lemma.

**Lemma 0:** For each \(n\) the set \(\{u_+(x, n, k), k \in I\}\), where \(I\) is an open interval in \(k^2 > (n + \alpha)^2\) determines \(u_+(x, n, k)\) for \(k \in \mathbb{R} \setminus S\). Thus the true distorted plane waves determine the generalized distorted plane waves.

The following observation is the main step in the proof.

**Lemma 1.** Letting \((L - (k - i0)^2)^{-1}g\) denote \(\lim_{\epsilon \to 0^-} (L - (k + i\epsilon)^2)^{-1}g\), the “incoming” solution, we have

\[
\int_{0}^{2\pi} T(x_1)u_+(x_1, T, m, k)dx_1 = \int_{D \cap \{0 < x_1 < 2\pi\}} e^{i((m+\alpha)x_1-x_2\lambda_0(k))(L-k^2)}\psi(L-(k-i0)^2)^{-1}(f\delta_T)dx,
\]

where \(\delta_T(\phi) = \int \phi(x_1, T)dx_1\).

Proof: We have

\[
u_+ = \psi e^{i\ell \cdot x} - (L - (k + i0)^2)^{-1}(L - k^2)(\psi e^{i\ell \cdot x}).\]

Hence, letting \(D_0 = D \cap \{(x_1, x_2) : 0 < x_1 < 2\pi\}\), for any smooth \(g\) satisfying (1) with bounded support in \(x_2\)

\[
\int_{D_0} u_+gdx = \int_{D_0} \psi e^{i\ell \cdot x}gdx - \int_{D_0} g(L-(k+i0)^2)^{-1}(L-k^2)(\psi e^{i\ell \cdot x})dx.
\]

Since \((L-(k-i0)^2)^{-1}\) is the adjoint of \((L-(k+i0)^2)^{-1}\), we have

\[
\int_{D_0} g(L-(k+i0)^2)^{-1}(L-k^2)(\psi e^{i\ell \cdot x})dx = \int_{D_0} (L-(k-i0)^2)^{-1}g(L-k^2)(\psi e^{i\ell \cdot x})dx.
\]

\(^1\)Note that case ii) can occur. Choose \(V\) with compact support so that \(-\partial_{x_2}^2 + V(x_2)\), considered as a Schrödinger operator on \(\mathbb{R}\), has a bound state, \(u \in L^2(\mathbb{R})\), i.e. \((-\partial_{x_2}^2 + V(x_2))u = Eu\). Then, taking \(m\) large enough that \((m+\alpha)^2 + E - V\) is strictly positive, defining \(\psi^{-1}(x) = ((m+\alpha)^2 + E)((m+\alpha)^2 + E - V)^{-1}\) and \(\psi = \exp(i(m+\alpha)x_1)u(x_2)\), we have \((1/\epsilon(x))\Delta \psi + ((m+\alpha)^2 + E)\psi = 0\).
Since $L = -\Delta$ on the support of $\psi$, for any smooth $h$ satisfying (2)

$$\int_{D_0} \overline{g}(L - k^2)(\psi e^{il x})dx = \int_{D_0} e^{il x}(2\nabla \psi \cdot \nabla + \Delta \psi)h dx. \quad (7)$$

Beginning with (5) and using (6) and (7), we have

$$\int_{D_0} u_+ g dx = \int_{D_0} \psi e^{il x} \overline{g} dx - \int_{D_0} (L - (k - i0)^2)^{-1}g(L - k^2)(\psi e^{il x}) dx =$$

$$\int_{D_0} \psi e^{il x} \overline{g} dx + \int_{D_0} (\nabla \psi \cdot \nabla + \Delta \psi)(L - (k - i0)^2)^{-1}g e^{il x} dx =$$

$$\int_{D_0} e^{il x}(L - k^2)\psi(L - (k - i0)^2)^{-1}g dx$$

Now approximating $f(x_1)\delta_T$ by $g$ of the form above gives (4).

With (4) we can easily prove

**Lemma 2:** Assume that $k \in \mathbb{R} \setminus (S \cup S_T)$ is fixed, where $S_T$ is the set of $k$ for which there are nontrivial solutions to $Lu - k^2 u = 0$ which vanish on $x_2 = T$ and are square-integrable on $D \cap \{x_2 < T\}$. Then the linear span of $\{u_+(x_1, T, m, k), m \in \mathbb{Z}\}$ is dense in $L^2(0 < x_1 < 2\pi)$.

**Proof:** Suppose that $f \in L^2(0 < x_1 < 2\pi)$ is orthogonal to the span of $\{u_+(x_1, T, m, k), m \in \mathbb{Z}\}$. Then (4) implies

$$\int_{-\infty}^{\infty} \left( \int_{0}^{2\pi} e^{i[(m+\alpha)x_1 - x_2 \lambda_0(k)]}(-\Delta - k^2)\psi(L - (k - i0)^2)^{-1}(f \delta_T)dx_1 \right)dx_2 = 0,$$

for all $m \in \mathbb{Z}$. Let

$$w = \text{def} (L - (k - i0)^2)^{-1}(f \delta_T).$$

Since $w$ is incoming, we have $\psi w = G_-(k)(L - k^2)\psi w$. Moreover, we have

$$(-\Delta - k^2)\psi w = f \delta_T - (2\nabla \psi \cdot \nabla + \Delta \psi)(L - (k - i0)^2)^{-1}f \delta_T = 0$$

for $x_2 > T$. So when we represent $\psi w$ as $G_-(k)(L - k^2)\psi w$ using the analog of (3) for $G_-(k)$, the integrand is supported in $y_2 \leq T$. Therefore, when $x_2 > T$, $|x_2 - y_2| = x_2 - y_2$ on the support of the integrand in (3), and the identity above implies

$$\psi w(x) = 0$$

for $x_2 > T$, i.e. $w(x) = 0$ for $x_2 > T$.

At this point the arguments for Case 1 and Case 2 separate. In Case 1, $w$ is an incoming solution to the homogeneous problem $(L - k^2)w = 0$ on $D \cap \{x_2 < T\}$, satisfying (1) and $w(x_1, T) = 0$. Thus $w = 0$ on all of $\partial(D \cap \{x_2 < T\})$. In this case we have for $R$ sufficiently large

$$0 = \int_{D \cap \{0 < x_1 < 2\pi\} \cap \{-R < |x_2| < R\}} ((Lw - k^2w) w - \pi (Lw - k^2w))a(x) dx =$$
\[ \int_0^{2\pi} w \frac{\partial w}{\partial x_2} - \pi \frac{\partial w}{\partial x_2} dx_1 |_{x_2 = -R} = \sum_{(m+\alpha)^2 < k^2} 2\pi i \sqrt{k^2 - (m + \alpha)^2} |a_m|^2 + O(e^{-\delta R}). \]  

(8)

The last equality comes from the representation of \( \psi w \) as \( G_{-}(k)((L - k^2)\psi w) \), i.e. for \( x_2 < -T \)

\[ w = \psi w = \sum_{\{m: (m+\alpha)^2 < k^2\}} e^{i(m+\alpha)x_1 - i x_2 \sqrt{k^2 - (m+\alpha)^2}} a_m + O(e^{-\delta|x_2|}). \]

From (8) it follows that the coefficients \( a_m \) of the propagating modes in \( w \) vanish, and \( w \in L^2(D \cap \{0 < x_1 < 2\pi\} \cap \{x_2 < T\}) \). In other words \( w \) is a Dirichlet eigenfunction for \( L \) in \( D \cap \{x_2 < T\} \) with the periodicity condition (2).

In Case 2 the situation is simpler. In this case one sees immediately that \( (L - (k - i\delta)^2)^{-1}(f_{\delta T}) \) is an eigenfunction for \( L \) on \( D \cap \{x_2 < T\} \), satisfying (2), and the proof is complete.

Let \( \Lambda(k) \) denote the Dirichlet-to-Neumann operator

\[ \Lambda(k) h = \frac{\partial u}{\partial x_2} \text{ on } x_2 = T, \]

where \( u \) is the outgoing solution to the boundary value problem \( Lu - k^2 u = 0 \) in \( D \cap \{x_2 < T\} \), \( u = h \) on \( x_2 = T \). Solutions to \( Lu - k^2 u = 0 \) which vanish on \( x_2 = T \) and are square-integrable on \( D \cap \{0 < x_1 < 2\pi\} \cap \{x_2 < T\} \) are eigenfunctions of \( L \) on \( D \cap \{x_2 < T\} \) with the periodicity condition (1). When \( k^2 \in S_T \), the set of eigenvalues for \( L \) on \( D \cap \{x_2 < T\} \), the Dirichlet-to-Neumann map with data on \( x_2 = T \) is not defined. Since the line \( x_2 = T \) is chosen more or less arbitrarily, for a fixed \( k \) one move \( k^2 \) out of \( S_T \) simply by shifting \( T \). The set \( S \), however, is intrinsic to the problem.

If the generalized distorted plane wave \( u_+(x, k, m) \) is known for \( x_2 > T \), then we know \( \partial u/\partial x_2 \) on \( x_2 = T \), and Lemma 2 has the following corollary.

**Corollary 1.** The set of generalized distorted plane waves \( \{u_+(x, k, m), m \in \mathbb{Z}\} \) for fixed \( k \in \mathbb{R} \setminus (S_T \cup S) \), determine \( \Lambda(k) \) on \( x_2 = T \).

We want to recover \( \Lambda(k) \) from the propagating modes. That follows easily at this point.

**Lemma 3.** The scattering data from propagating modes,

\[ \{ a_m(n, k) : (m + \alpha)^2 < k^2 \text{ and } (n + \alpha)^2 < k^2 \} \]

given for all \( k \in \mathbb{R} \setminus S \), determine the distorted plane waves in \( x_2 \geq T \).

Proof: By (2)

\[ a_m(n, k) = \frac{e^{-ix_2\sqrt{\lambda_m(k)}}}{2\pi} \int_0^T e^{-ix_1(m + \alpha)} u_+(x_1, T, k, n) dx_1, \]

it follows that \( a_m(n, k) \) is analytic in \( k \) on the set where \( u_+ \) is analytic in \( k \). For fixed \( m \) and \( n \), \( a_m(n, k) \) will be part of the scattering data from propagating modes when
$k$ is sufficiently large. Thus for each $m$ and $n$ the scattering data from propagating modes determine $a_m(n, k)$ on $\mathbb{R}\setminus S$. Thus by (2) the propagating modes determine the generalized distorted plane waves.

Combining Corollary 1 and Lemma 3, we conclude the the propagating modes determine $\Lambda(k)$ for $k \in \mathbb{R}\setminus (S \cup S_T)$.

§3. Determination of the Dirichlet-to-Neumann Map for Wave Guides

The arguments of the preceding section apply to the wave guides with modifications that we give here.

Since now the unperturbed operator is $-c_0^2\Delta$, we need to replace (2) with a representation for the outgoing fundamental solution for $-c_0^2\Delta$. To obtain this representation we separate variables and use expansion in the eigenfunctions (chosen to be real-valued) of the Sturm-Liouville problem

$$\phi''_m(x_1, k) + \frac{k^2}{c_0^2(x_1)}\phi_m(x_1, k) = \mu_m(k)\phi_m(x_1, k)$$

with $\phi_m(0) = 0$, $\phi'_m(c, k) = 0$. Using this basis and assuming that $k$ is chosen so that $\mu_m(k) \neq 0, m \in \mathbb{N}$, one checks that for $f$ with bounded support in $x_2$

$$u(x, k) = \sum_{m=1}^{\infty} \frac{1}{2\sqrt{\mu_m(k)}} \int_{[0, \Delta] \times \mathbb{R}} e^{i \sqrt{\mu_m(k)}|x_2-y_2|} \phi_m(x_1, k)\phi_m(y_1, k)\frac{f(y)}{c_0(y)} dy,$$

is a solution to $(L-k^2)u = f$ when $k$ is real. To see that this is the outgoing solution we will show that $u(x, k)$ continues to a square-integrable solution when $k$ moves into the upper half plane. Since the boundary conditions make $d^2/dx_1^2 + k^2/c_0^2(x_1)$ self-adjoint when $k$ is real, the functions $\phi_m(x_1, k)$ and $\mu_m(k)$ are analytic in $k$ by Rellich’s theorem. This is an elementary result here, since $\mu_m(k)$ is a simple eigenvalue when $k$ is real. Thus for $\epsilon > 0$, if we can show that $\text{Im}\{\mu_m(k+i\epsilon)\} > 0$ when $\mu_m(k) > 0$, the choice of $\sqrt{\mu_m(k+i\epsilon)}$ that we use here (see the definitions preceding Theorem 1 in §1) will make $\text{Im}\{\sqrt{\mu_m(k+i\epsilon)}\} > 0$. However, this follows immediately from the observation that $d\mu_m(k)/dk > 0$ for $k$ real. Thus we conclude that for all $f$ for which (9) is a finite sum, $u$ extends to a square-integrable solution to $(L-k^2)u = f$ as $k$ moves into the upper half-plane. Thus, on the complement of the thresholds the operator $G_+(k)$, defined by

$$G_+(k)f = \sum_{m=1}^{\infty} \phi_m(x_1, k) \int_{[0, \Delta] \times \mathbb{R}} e^{i \sqrt{\mu_m(k)}|x_2-y_2|} \phi_m(y_1, k)\frac{f(y)}{c_0(y)} dy,$$

coincides with the limit of $(-c_0^2\Delta - k^2I)^{-1}$ as $\text{Im}\{k\} \to 0_+$ on a dense set of $f$. Since an easy limiting absorption argument shows that $\lim_{k \to \infty} (-c_0^2\Delta - k^2I)^{-1}f$ exists for $f$ with bounded support, it follows that $G_+(k)$ is the outgoing fundamental solution. The same construction, replacing the square roots in (9) with their complex conjugates, leads to the incoming fundamental solution $G_-(k)$.

As stated in §1, distorted plane waves for the wave guide are obtained by solving $(L-k^2)u = 0$ with the given boundary conditions on $x_1 = 0$ and $x_1 = \Delta$ for $u = \Phi(x, k, m) + v_\epsilon$, where $\Phi$ is a generalized eigenfunction for $-c_0^2(x_1)\Delta$, i.e.
\( \Phi(x, m, k) = \exp(ix_2 \sqrt{\mu_m(k)}) \phi_m(x_1, k) \). Note that for this to be a true distorted plane wave \( \mu_m(k) \) should be positive. However, as in §2 we allow “generalized” distorted plane waves where \( \mu_m(k) < 0 \). As in §2 the construction of outgoing distorted plane waves \( u_+(x, m, k) = \Phi(x, m, k) + v_+(x, m, k) \) is done by limiting by absorption. As in §2, \( u_+ \) has a representation \( u_+ = \psi(x_2) \Phi + v_+ \) where

\[
v_+ = - \lim_{\epsilon \to 0^+} (L - (k + i\epsilon)^2)^{-1}(L - k^2)(\psi(x_2)\Phi),
\]

with \( L = -c^2 \Delta \). Here the cutoff function \( \psi \in C^\infty(\mathbb{R}) \) satisfies \( \psi \equiv 1 \) for \( |x_2| > T + 1 \) with support contained in \( |x_2| > T \). As before, the functions \( u_\pm \) do not depend on the choice of \( \psi \). Moreover, the exceptional set \( S \) is again the union of the thresholds and the set of \( k \) for which there is a nontrivial, square-integrable solution to \( (L - k^2)u = 0 \) in \( [0, B] \times \mathbb{R} \). The generalized distorted plane waves \( u_+ \) have analytic continuations to \( \text{Im}\{k\} > 0 \), and hence as in the case of gratings, \( \{u_+(x, m, k), k \in I\} \), where \( I \) is an open interval in \( \{k : \mu_m(k) > 0\} \) determines \( u_+(x, m, k) \) for \( k \in \mathbb{R} \setminus S \) (note that \( \mu_m(k) > 0 \) for \( k \) sufficiently large for each \( m \)). In other words the generalized distorted plane waves are again determined by the true distorted plane waves via analytic continuation.

The analog of (4) for wave guides is

\[
\int_0^B f(x_1) u_+(x_1, T, m, k) dx_1 =
\int_{[0, B] \times \mathbb{R}} \Phi(x, m, k) (L - k^2) \psi(L - (k - i0)^2)^{-1}(f_\delta_T) dx,
\]

and this identity shows that Corollary 1 holds for wave guides. Likewise we have the following analog of Lemma 3.

**Lemma 4.** The scattering data from propagating modes,

\( \{b_m(n, k) : \mu_n(k) > 0\} \) and \( \mu_m(k) > 0 \}

given for all \( k \in \mathbb{R} \setminus S \), determine the distorted plane waves in \( x_2 \geq T \).

Since (2') gives,

\[
b_m(n, k) = \frac{e^{-ix_2 \sqrt{\mu_m(k)}}}{2\pi} \int_0^T \phi_m(x_1, k) v_+(x_1, T, k, n) dx_1,
\]

it follows that \( a_m(n, k) \) is analytic in \( k \) on the set where \( u_+ \) is analytic in \( k \), the proof of Lemma 3 applies here, and again conclude that the propagating modes determine \( \Lambda(k) \) for \( k \in \mathbb{R} \setminus (S \cup S_T) \).

§4. **Reduction to the hyperbolic inverse problem**

We will begin with the wave guide problem. Consider the hyperbolic equation

\[
v_{tt} = c^2(x) \Delta v
\]
in \( \{ (x_1, x_2) \in [0, B] \times (\infty, T] \} \) with zero initial conditions, \( v(x, 0) = 0 \), \( v_t(x, 0) = 0 \), and the boundary conditions

\[
v(0, x_2, t) = 0, \quad \frac{\partial v}{\partial x_1}(B, x_2, t) = 0, \quad \text{and} \quad v(x_1, T, t) = g(x_1, t).
\]

Let \( \Lambda_H \) denote the hyperbolic Dirichlet-to-Neumann operator corresponding to this initial-boundary value problem:

\[
\Lambda_H g = \frac{\partial v}{\partial x_2} \quad \text{for} \quad (x_1, t) \in [0, B] \times [0, \infty).
\]

The following theorem is a particular case of results in [B] and [KKL] (see also [E1], [E2]).

**Theorem 2.** The hyperbolic Dirichlet-to-Neumann map, \( \Lambda_H \) on \( x_2 = T \), uniquely determines the sound speed \( c(x) \).

To deduce Theorem 1 from Theorem 2 we proceed as follows. Let \( \Lambda(k) \) be the elliptic Dirichlet-to-Neumann operator defined previously, for \( c^2(x) \Delta \), i.e. \( \Lambda(k)h = \partial u/\partial x_2 \) on \( x_2 = T \), where \( u \) is the outgoing solution to the boundary value problem

\[
c^2(x) \Delta u + k^2 u = 0 \quad \text{in} \quad [0, B] \times (-\infty, T], \quad u = h \quad \text{on} \quad x_2 = T
\]

with the Dirichlet boundary condition on \( x_1 = 0 \) and the Neumann condition on \( x_1 = B \). As we observed earlier, \( \Lambda(k) \) is analytic in \( k \) off the discrete set \( S_T \). Hence, using the Fourier-Laplace transform in \( t \), we can recover \( \Lambda_H \) from \( \Lambda(k) \), given for \( k_0 - \epsilon < k < k_0 + \epsilon \). Since we showed in §3 that the propagating modes determine \( \Lambda(k) \), this completes the proof of Theorem 1 for wave guides.

Since we have also shown for gratings that \( \Lambda(k) \) for \( k \in \mathbb{R} \setminus (S \cup S_T) \) is determined by scattering data from propagating modes, the only change in the argument needed to prove Theorem 1 for gratings is in the citation of results on the hyperbolic Dirichlet-to-Neumann map. Here the hyperbolic Dirichlet-to-Neumann operator, \( \Lambda_H \), is defined by

\[
\Lambda_H g = \frac{\partial v}{\partial x_2} \quad \text{on} \quad x_2 = T,
\]

where \( v \) is the solution to

\[
v_{tt} = Lv \quad \text{in} \quad D \cap \{ x_2 < T \} \times \{ 0 \leq t < \infty \}
\]

satisfying the periodicity condition (1) and the initial-boundary conditions \( v(x, 0) = v_t(x, 0) = 0 \) and \( v(x_1, T, t) = g \). In this setting the uniqueness results of [B], [KKL] and [E1,2] imply that that \( \Lambda_H \) given on \( x_2 = T \) determine both the coefficients of \( L \), i.e. the permittivity \( \epsilon(x) \), and the domain \( D \). This completes the proof of Theorem 1.

**Appendix I.**

The limiting absorption argument needed for the construction of distorted plane waves in §2 and §3 is particularly simple because the coefficients and domains of the perturbed and unperturbed operators are identical outside a compact set. Hence,
rather than sending the reader to references for more general situations we give this argument here.

In the case of gratings let $L$ be the operator in (TE) or (TM) considered as an operator in $L^2(D \cap \{0 \leq x_1 \leq 2\pi\}, a(x)dx)$ with domain the set of smooth functions of compact support satisfying (1) and, in the case of (TE), the Dirichlet condition on $\partial D$. In the case of wave guides $L = -i^2(x)\Delta$ in $L^2(\{x \in \mathbb{R}^2 : 0 \leq x_1 \leq B\}, e^{-2}dx)$ with domain the set of smooth functions of compact support satisfying the Dirichlet condition on $x_1 = 0$ and the Neumann condition on $x_1 = B$. We will assume in both cases that it is known that $L$ with these domains is essentially self-adjoint.

The essential features the operators $L$ in the preceding paragraph are:

a) $L$ is second order elliptic: $\|u\|_{H^2(D; \{|x_2| < T+1\})} \leq C\|Lu\|_{H^0(D; \{|x_2| < T+2\})}$, when $u$ is in the domain of $L$, and

b) $Lu = L_0u$ when $u$ is in the domain of $L$ with support in $|x_2| > T$, and $(L_0 - \lambda I)^{-1}$ has an explicit continuation from $\text{Im}\{\lambda\} < 0$ to the real axis minus the thresholds.

Let $\phi(x_2) \in C^\infty_0(|x_2| < T + 1)$ satisfy $\phi(x_2) = 1$ on an open neighborhood of $|x_2| \leq T$. Then, since $L$ is essentially self-adjoint, $(L - \lambda I)^{-1}$ exists for $\text{Im}\{\lambda\} \neq 0$, and by b) we can write

$$u = (L - \lambda I)^{-1}g = \phi(L - \lambda I)^{-1}g + (L_0 - \lambda I)^{-1}[(L_0 - \lambda I)((1 - \phi)u)]. \quad (A.1)$$

Note that this representation holds because $(1 - \phi)u$ is in the domain of $L_0$. Assuming that $g$ is supported in $|x_2| < T$, it follows that $(L_0 - \lambda I)((1 - \phi)u) = -(L_0, \phi|u)$ and hence is supported in $|x_2| < T + 1$.

Now let $\psi(x_2) \in C^\infty_0(|x_2| < T + 2)$ satisfy $\psi(x_2) = 1$ on an open neighborhood of $|x_2| \leq T + 1$, and consider the behavior of $w(x, \lambda) = \psi(L - \lambda I)^{-1}g$ as $\lambda$ approaches $\lambda_\infty$ on the real axis minus the thresholds. If $w$ remains bounded in $L^2$-norm, it follows from a) that we can extract a sequence $\{\lambda_j\}$ converging to $\lambda_\infty$ such that $w(x, \lambda_j)$ converges in $H^1(D)$. Thus $(L_0 - \lambda I)((1 - \phi)u)$ converges in $H^0(D)$, and both terms on the right in (A.1) converge, giving us a solution to $(L - \lambda_\infty I)u = g$ represented for $x_2$ large by the limit of $(L_0 - \lambda I)^{-1}$.

If the $L^2$-norm of $w(x, \lambda_j)$ is not bounded, then we can assume that it tends to infinity as $\lambda \to \infty$, and consider

$$v(x, \lambda_j) = \frac{1}{\|w(\cdot, \lambda_j)\|_{H^0}}w(x, \lambda_j).$$

Since the $L^2$-norm of $v(x, \lambda_j)$ remains bounded as $\lambda \to \infty$, the argument in preceding paragraph shows that $(1/\|w(\cdot, \lambda_j)\|_{H^0})u(x, \lambda_j)$ converges to a solution to $(L - \lambda_\infty I)u = 0$ represented for $x_2$ large by the limit of $(L_0 - \lambda I)^{-1}$. Moreover, this solution is non zero since $\|v\|_{H^0} = 1$. Thus we have the Fredholm alternative of limiting absorption: either there is a solution satisfying the appropriate radiation condition to $(L - \lambda_\infty I)u = g$ for all $g$ supported in $|x_2| < T$, or there is a solution to the homogeneous problem satisfying the radiation condition.

In §2 and §3 $\lambda = (k + i\epsilon)^2$ with $\epsilon > 0$. Thus $\lambda$ is approaching the real axis from the upper half-plane when $k > 0$ and from the lower half-plane when $k < 0$. When $k = 0$, $\lambda < 0$. However, this causes no difficulty because in all cases we consider $L$ is a nonnegative operator so that $(L - \lambda I)^{-1}$ exists when $\lambda < 0$, and the preceding argument applies.
References:


[B] Belishev, M., Boundary control in reconstruction of manifolds and metrics (the BC method), Inverse Problems 13(1997), R1-R45.


