Gliding Beams and Whispering Gallery Quasi-modes

We will carry out this construction for the constant coefficient wave equation \( u_{tt} = \Delta u \) in a bounded domain \( \Omega \) with smooth, strictly convex boundary \( \partial \Omega \). It can also be done for general second order hyperbolic equations in domains where the convexity only holds along the ray that one follows.

Consider geodesic normal coordinates \((x, y) = (x, y_1, \ldots, y_d)\) near \( \partial \Omega \), where \( x = 0 \) on \( \partial \Omega \) and \( x > 0 \) in \( \Omega \). Hence near \( \partial \Omega \), the principal part of \( \Delta \) in these coordinates becomes,

\[
\partial_x^2 + \sum_{1 \leq j, k \leq d} g_{jk}(x, y) \partial_{y_j} \partial_{y_k},
\]

and

\[
ds^2 = \sum_{1 \leq j, k \leq d} g_{jk}(0, y) dy_j dy_k, \quad (g_{jk}) = (g^{jk})^{-1}
\]
is the metric induced on \( \partial \Omega \) by the Euclidean metric on \( \mathbb{R}^{d+1} \). Our gliding beam will follow a ray \( \gamma \) in space-time defined as follows. Let \((y(s), \eta(s))\) be a solution to Hamiltonian equations

\[
\dot{y} = H_y(y, \eta), \quad \dot{\eta} = -H_\eta(y, \eta), \quad H = \eta \cdot G(0, y) \eta, \quad G(x, y) = (g^{jk}(x, y)).
\]

This implies that \( y(s) \) is a geodesic in \( \partial D \) for the induced metric. We will assume that \( H(y(s), \eta(s)) \equiv 1 \), and consequently \( s/2 \) is arc-length. We define \( y_c(t) = y(t/2) \), and refer to \( \gamma \), parameterized by \((y_c(t), t)\), as the “central ray” in what follows.

We will construct asymptotic solutions to the boundary value problem

\[
0 = \square u \overset{\text{def.}}{=} u_{tt} - \Delta u \text{ on } \Omega \times \mathbb{R}_t, \quad u = 0 \text{ on } \partial \Omega \times \mathbb{R}_t,
\]

concentrated near the curve \( \gamma \) in \( \partial D \). To construct these solutions we will use the following Ansatz for \( k >> 0 \):

\[
u(x, t; k, \alpha) = e^{ik\phi}[a \text{Ai}(k^{2/3} \rho) + i k^{-1/3} b \text{Ai}'(k^{2/3} \rho)].
\]  

(1)

Here \( \text{Ai}(s) \) is the Airy function defined by

\[
\text{Ai}(s) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i(\xi s + \xi^3/3)} d\xi,
\]

and \( \text{Ai}'(s) \) is its derivative. Note that \( \text{Ai}''(s) - s \text{Ai}(s) = 0 \). The phase \( \phi = \phi(x, y, t, \alpha) \) is complex-valued with nonnegative imaginary part. The “auxiliary phase”, \( \rho = \rho(x, y, t, \alpha) \), satisfies \( \rho(0, y, t, \alpha) = \alpha \). The coefficients \( a \) and \( b \) are asymptotic in inverse powers of \( k \):

\[
a(x, y, t, \alpha) \sim \sum_{n=0}^{\infty} \frac{a_k(x, y, t, \alpha)}{k^n}, \quad b(x, y, t, \alpha) \sim \sum_{n=0}^{\infty} \frac{b_k(x, y, t, \alpha)}{k^n}.
\]

In order to have \( u = 0 \) on \( \partial \Omega \times \mathbb{R}_t \) we require \( b(0, y, t, \alpha) \equiv 0 \) and \( \alpha = \alpha(k) = k^{-2/3} z \), where \( \text{Ai}(z) = 0 \). The choice of \( z \) closest to 0 is \( z_1 = -2.33811 \ldots \).
In Euclidean coordinates \((z_1, \ldots, z_{d+1})\) on \(\mathbb{R}^{d+1}\) the \(u\) given by (1) satisfies
\[
u_{tt} - \Delta u = e^{ik\phi}[(k^2 E_2 + kE_1 + E_0)Ai(k^2/\rho) + (k^{5/3} F_2 + k^{2/3} F_1 + k^{-1/3} F_0)Ai'(k^2/\rho)],
\]
where
\[
E_2 = [-\phi_t^2 + |\nabla_z \phi|^2 + \rho(\rho_t^2 - |\nabla_z \rho|^2)]a - \rho[\rho_t \phi_t - \nabla_z \rho \cdot \nabla_z \phi]b,
\]
\[
E_1 = i[2\phi_t a_t - 2\nabla_z \phi \cdot \nabla_z a + (\Box \phi) a + \rho(2\rho_t b_t - 2\nabla_z \rho \cdot \nabla_z b + (\Box \rho) b) + (\rho_t^2 - |\nabla_z \rho|^2) b],
\]
\[
E_0 = \Box a,
\]
\[
F_2 = [\phi_t^2 - |\nabla_z \phi|^2]b + i\rho[\rho_t \phi_t - \nabla_z \rho \cdot \nabla_z \phi]a,
\]
\[
F_1 = i[2\phi_t b_t - 2\nabla_z \phi \cdot \nabla_z b + (\Box \phi) b] + [2(\rho_t a_t - \nabla_z \rho \cdot \nabla_z a) + (\Box \rho) a], \quad \text{and}
\]
\[
F_0 = i\Box b.
\]
As we work with these equations we will need to rewrite them in the \((x, y)\)-coordinates. In particular, in order to have \(E_2 = F_2 = 0\) we need the eichonal equations in \((x, y)\)-coordinates
\[
\phi_t^2 - \phi_x^2 - \phi_y \cdot G \phi_y - \rho(\rho_t^2 - \rho_x^2 - \rho_y \cdot G \rho_y) = 0, \quad (2a)
\]
and
\[
\rho_t \phi_t - \rho_x \phi_x - \rho_y \cdot G \phi_y = 0. \quad (2b)
\]
To make \(kE_1 + E_0 = O(k^{-\infty})\) and \(kF_1 + F_0 = O(k^{-\infty})\) we need the transport equations in \((x, y)\)-coordinates
\[
2ki[\phi_t a_t - \phi_x a_x - \phi_y \cdot Ga_y + \frac{1}{2}(\Box \phi) a + \rho(\rho_t b_t - \rho_x b_x - \rho_y \cdot G \rho_y) + \frac{1}{2}(\Box \rho) b] + \frac{1}{2}(\Box \rho) b + \frac{1}{2}(\rho_t^2 - \rho_x^2 - \rho_y \cdot G \rho_y) b + i\Box a = O(k^{\infty}), \quad (3a)
\]
\[
-2ki[\phi_t b_t - \phi_x b_x - \phi_y \cdot G b_y + \frac{1}{2}(\Box \phi) b + (\rho_t a_t - \rho_x a_x - \rho_y \cdot Ga_y + \frac{1}{2}(\Box \rho) a) + i\Box b = O(k^{\infty}). \quad (3b)
\]
The solution of these equations with the constraint \(\rho - \alpha = O(\alpha^{\infty})\) on \(\partial \Omega\) is fairly involved. It is due to D. Ludwig (CPAM 20(1967), and it is discussed in detail in M.E.Taylor’s \textit{Pseudo-differential Operators}, Chapter X. However, since we are constructing gaussian beam approximations, the equations (2a),(2b),(3a) and (3b) reduce to systems of ODEs along \(\gamma\). We require that \(\phi\) and \(\rho\) are real-valued on \(\gamma\) and the Hessian of imaginary part of \(\phi\) on \(x = 0\) is positive definite in directions transverse to \(\gamma\). Since \(Ai(s)\) is exponentially decreasing in the sector \(|\text{Im}\{s\}| < \sqrt{3}\text{Re}\{s\}\), these conditions on \(\phi\) and \(\rho\) make \(u = O(k^{\infty})\) away from \(\gamma\). Hence, it will suffice to construct the Taylor series of \(\phi\), \(\rho\), \(a\) and \(b\) in \((x, y, t, \alpha)\) along \(\gamma\) so that (2ab) and (3ab) hold to high order on \(\gamma\). This is significantly simpler than the full solution of the eichonal and transport equations.

The constraint \(\rho - \alpha = O(\alpha^{\infty})\) on \(\partial \Omega\) implies that we can assume \(\rho(0, y, t, \alpha) \equiv \alpha\) in our computations of derivatives. The calculations that follow are determined by that constraint combined with the requirement
\[
\rho_x(0, y, t, 0, \alpha) > 0 \quad (4)
\]
In view of (4) and $\rho(0, y, t, \alpha) = \alpha$, it follows from (2b) that $\phi_x(0, y, t, \alpha) \equiv 0$ as well. Thus on $x = \alpha = 0$ equation (2a) becomes

$$\phi_t^2 - \phi_y \cdot G\phi_y = 0.$$  

To make this equation hold to infinite order in $(y, t)$ on $(y(t), t)$ we construct $\phi(0, y, t, 0)$ as the phase of a gaussian beam concentrated on $(y(t), t)$. See, for instance, http://www.math.ucla.edu/ ralston/pub/Gaussnotes.pdf for this.

Differentiating (2a) with respect to $x$ and evaluating on $x = \alpha = 0$ gives

$$[\phi_y \cdot G_x\phi_y](0, y, t, 0) - (\rho_x(0, y, t, 0))^3 = 0.$$  

This is where the the strict convexity of $\partial \Omega$ enters: since $\phi_y(0, y(t), t, 0) = \eta(t) = -G^{-1}y_c'(t)$, formula (5) implies

$$- \sum_{1 \leq j, k \leq d} \frac{\partial g_{jk}}{\partial x}(0, y(t))(y_c)'_j(t)(y_c)'_k(t) - (\rho_x(0, y(t), t, 0))^3 = 0.$$  

The strict convexity assumption implies that

$$\sum_{1 \leq j, k \leq d} \frac{\partial g_{jk}}{\partial x}(0, y(t))\dot{y}_j(t)\dot{y}_k(t) < 0$$  

(curves get shorter as $x$ increases). So (5) implies $\rho_x(0, y(t), t, 0) > 0$, and, since $\alpha$ will be close to 0, that gives (4).

Now we need to become more systematic. Following Ludwig and Taylor, we will determine $\phi$, $\rho$, $a$ and $b$ as formal power series in $\alpha$

$$\phi(x, y, t, \alpha) = \sum_{n=0}^{\infty} \phi^{(n)}(x, y, t)\alpha^n \quad \rho(x, y, t, \alpha) = \sum_{n=0}^{\infty} \rho^{(n)}(x, y, t)\alpha^n$$

$$a(x, y, t, \alpha; k) = \sum_{n=0}^{\infty} a^{(n)}(x, y, t; k)\alpha^n \quad b(x, y, t, \alpha; k) = \sum_{n=0}^{\infty} b^{(n)}(x, y, t; k)\alpha^n.$$  

We will substitute these formal power series into (2ab) and (3ab) and equate the coefficients of each power of $\alpha$ to zero, starting with the constant terms in the series. We have the following key facts

(A) $\rho^{(n)}(0, y, t) \equiv 0$ for $n \neq 1$, $\rho^{(1)}(0, y, t) \equiv 1$ and $b^{(n)}(0, y, t) \equiv 0$ for all $n$.

(B) $\rho_x(0, y, t) > 0$.

With (A) and (B) we can determine the Taylor series of $\phi$, $\rho$, $a$ and $b$ in $x$ about $x = 0$. This proceeds as follows. For all $n > 1$ the results of setting the coefficient of $\alpha^n$ to zero in (2a), (2b), (3a) and (3b) respectively are

$$2[\phi_t^{(0)}\phi_t^{(n)} - \phi_x^{(0)}\phi_x^{(n)} - \phi_y^{(0)} \cdot G\phi_y^{(n)} - \rho_t^{(0)}\rho_t^{(n)} - \rho_x^{(0)}\rho_x^{(n)} - \rho_y^{(0)} \cdot G\rho_y^{(n)}]$$

$$-\rho^{(n)}((\rho_t^{(0)})^2 - (\rho_x^{(0)})^2 - \rho_y^{(0)} \cdot G\rho_y^{(0)}) = F_n.$$  

(6a)
\( \phi_t^{(n)} \rho_t^{(n)} + \phi_t^{(n)} \rho_t^{(0)} - \phi_x^{(n)} \rho_x^{(n)} - \phi_x^{(n)} \rho_x^{(0)} - \phi_y^{(n)} \rho_y^{(n)} - \phi_y^{(n)} \rho_y^{(0)} = G_n \) (6b)

\[
2ik[\phi_t^{(0)} a_t^{(n)} - \phi_x^{(0)} a_x^{(n)} - \phi_y^{(0)} \cdot G a_y^{(n)} + \frac{1}{2} \Box a^{(n)}] + \rho^{(0)}(\rho_t^{(0)} b_t^{(n)} - \rho_x^{(0)} b_x^{(n)} - \rho_y^{(0)} \cdot G b_y^{(n)} + \frac{1}{2} \Box b^{(n)})
+ \frac{1}{2}(\rho_t^{(0)})^2 - (\rho_x^{(0)})^2 - (\rho_y^{(0)} \cdot G \rho_y^{(0)})b^{(n)}] + \Box a^{(n)} = H_n + O(k^n) \] (7a)

\[
2k[\phi_t^{(0)} b_t^{(n)} - \phi_x^{(0)} b_x^{(n)} - \phi_y^{(0)} \cdot G b_y^{(n)} + \frac{1}{2} \Box b^{(n)}] + \rho_t^{(0)}a_t^{(n)} - \rho_x^{(0)} a_x^{(n)} - \rho_y^{(0)} \cdot G a_y^{(n)} + \frac{1}{2} \Box a^{(n)} - i \Box a^{(n)} = I_n + O(k^n) \] (7b)

Here \( F_n \) and \( G_n \) are determined by \( \phi^{(j)} \) and \( \rho^{(j)} \), \( 0 \leq j < n \), and \( F_0 = G_0 = 0 \). Similarly, \( H_n \) and \( I_n \) are determined by \( \phi \) and \( \rho \), and by \( a^{(j)} \) and \( b^{(j)} \), \( 0 \leq j < n \). Here also \( H_0 = I_0 = 0 \). For \( n = 0 \) instead of (6ab) and (7ab) one simply gets (2ab) and (3ab) with \( \phi, \rho, a \) and \( b \) replaced by \( \phi^{(0)}, \rho^{(0)}, a^{(0)} \) and \( b^{(0)} \).

Now we can describe the procedure for determining the Taylor series in \( x \). Let \( \partial_x^p(M) \) denote the equation obtained by taking differentiating equation (M) \( p \) times with respect to \( x \) and evaluating on \( x = 0 \). For \( p = 0, 1, \ldots \) we will determine \( \partial_x^{p+1}\phi^{(n)}(0, y, t) \) from \( \partial_x^p(6b) \). In particular, for \( p = n = 0 \) we have \( \phi_x^{(0)}(0, y, t) \equiv 0 \). Hence, in view of (A) and (B), for all \( n \) the coefficient of \( \partial_x^{p+1}\phi^{(n)}(0, y, t) \) in \( \partial_x^p(6b) \) is \( \rho_x^{(0)}(0, y, t) \) which is positive by (B), and all the other terms in \( \partial_x^p(6b) \) are determined by \( \partial_x^l \phi^{(l)} \) and \( \partial_x^l \rho^{(l)} \) with \( l \leq n \) and \( j < p \) when \( l = n \).

Since \( \phi_x^{(0)}(0, y, t) = 0 \), evaluating (6a) on \( x = 0 \) gives

\[
2[\phi_t^{(0)} \phi_t^{(n)} - \phi_y^{(0)} \cdot G \phi_y^{(n)}] = F_n(0, y, t) \text{ for } n > 1, \text{ and } (8)
\
2[\phi_t^{(0)} \phi_t^{(1)} - \phi_y^{(0)} \cdot G \phi_y^{(1)}] + (\rho_x^{(0)})^2 = F_n(0, y, t) \text{ for } n = 1 \quad (8')
\]

Differentiating (2a) with respect to \( \alpha \) and evaluating on \( x = 0 \), one sees that \( F_1(0, y, t) = 0 \). Since we only require that (8) and (8') hold to all orders on \( (0, y(t), t) \), these equations reduce to systems of ODE’s on \( (0, y(t), t) \) that are discussed in the gaussian beam construction. In particular, using (5), (8') reduces to

\[
2\phi' + (\phi_y \cdot G_x \phi_y)^{2/3} = 0, \quad (9)
\]

where \( f' = (d/dt)f(y_c(t), t) \). Further computation shows that

\[
|\phi_y \cdot G_x \phi_y| = 2\kappa(t), \quad (10)
\]

where \( \kappa(t) \) is the curvature of \( \gamma \) at \( y(t) \). This will be used later in the formulas for quasi-modes.

\[\begin{aligned}
1 \text{As noted above, one really has } \phi_x^{(n)}(0, y, t) \equiv 0 \text{ for all } n, \text{ but that is not needed here.}
\end{aligned}\]
In \( \partial_x^p (6a) \) the coefficient of \( \partial_x^p \rho^{(n)} \) is \( 2(\rho_x^{(0)})^2 \) when \( n > 0 \), \( (p + 3)(\rho_x^{(0)})^2 \) when \( n = 0 \) and \( p \geq 2 \), and \( (\rho_x^{(0)})^2 \) when \( n = 0 \) and \( p = 1 \). All other terms in \( \partial_x^p (6a) \) are determined by \( \partial_x^l \phi^{(l)} \) and \( \partial_x^{j-1} \rho^{(l)} \) with \( l \leq n \) and \( j \leq p \) when \( l = n \). So first using \( \partial_x^p (6b) \) and then \( \partial_x^{p+1} (6a) \) for \( p = 0, 1, \ldots \), we determine the Taylor series in \( x \) of \( \phi^{(n)} \) and \( \rho^{(n)} \) about \( x = 0 \) for all \( n \).

The procedure for the transport equations is almost identical with \( a \) and \( b \) playing the roles of \( \phi \) and \( \rho \), respectively. Since \( a^{(n)} \) and \( b^{(n)} \) are themselves asymptotic series in \( k \), if one wrote out all the details, would be done successively in inverse powers of \( k \), first determining \( a_0^{(n)} \) and \( b_0^{(n)} \), then \( a_1^{(n)} \) and \( b_1^{(n)} \), etc. We will suppress that here. However, in what follows we will consider \( \Box a^{(n)} \) in (7a) and \( \Box b^{(n)} \) in (7b) as known terms, since their contributions will always be determined by preceding terms in the asymptotic series in \( k \).

Conditions (A) and (B) plus \( \phi_x^{(0)} (0, y, t) \equiv 0 \) imply that on \( x = 0 \)

\[
2ik[\phi_t^{(0)} a_t^{(n)} - \phi_y^{(0)} \cdot Ga_y^{(n)} + \frac{1}{2} \Box \phi^{(0)} a^{(n)}] + \Box a^{(n)} = H_n(0, y, t) + O(k) \infty
\]

This is an (inhomogeneous) transport equation associated with the phase \( \phi^{(0)} \), and under our assumptions it reduces to ODE’s along \( (0, y(t), t) \). Assuming that those have been solved – note that we can prescribe the Taylor series in \( y \) of \( a^{(n)}(0, y, 0) \) about \( y(0) \) – we can assume that \( a^{(n)}(0, y, t) \) is known, given \( \phi, \rho \), and \( a^{(j)} \) and \( b^{(j)} \) for \( j < n \).

Now, imitating the previous argument, for \( p = 0, 1, \ldots \) we will determine \( \partial_x^{p+1} a^{(n)}(0, y, t) \) from \( \partial_x^p (7b) \). Conditions (A) and (B) imply that for all \( n \) the coefficient of \( \partial_x^{p+1} a^{(n)}(0, y, t) \) in \( \partial_x^p (7b) \) is \( -\rho_x^{(0)}(0, y, t) \) and all the other terms in \( \partial_x^p (7b) \) are determined by \( \phi, \rho, \partial_x^l \phi^{(l)} \) and \( \partial_x^j \rho^{(l)} \) with \( l \leq n \) and \( j < p \) when \( l = n \). In \( \partial_x^p (7a) \) the coefficient of \( \partial_x^p b^{(n)} \) is \( -3ik(\rho_x^{(0)})^2 \) when \( n > 0 \) and \( p \geq 1 \). All other terms in \( \partial_x^p (7a) \) are determined by \( \phi, \rho, \partial_x^l \phi^{(l)} \) and \( \partial_x^j \rho^{(l)} \) with \( l \leq n \) and \( j < p \) when \( l = n \). So first using \( \partial_x^p (7b) \) and then \( \partial_x^{p+1} (7a) \) for \( p = 0, 1, \ldots \), we determine the Taylor series in \( x \) of \( a^{(n)} \) and \( b^{(n)} \) about \( x = 0 \) for all \( n \). This completes the construction of the asymptotic solution \( u(x, y, t; k) \) for this problem.

**Remark on Quasi-modes**

The preceding construction can be adapted to produce quasi-modes for the “whispering gallery” problem. Assume that \( \gamma \) is a periodic geodesic in \( \partial D \) with the metric induced from the Euclidean metric on \( \mathbb{R}^{d+1} \) that is stable in the following sense: the eigenvalues of the (linearized) Poincaré map is a direct sum of rotations through distinct angles \( \theta_1, \ldots, \theta_{d-1} \), satisfying \( 0 < \theta_1 < 2\pi \), \( \theta_i \neq 2\pi - \theta_j \) for all \( i, j \). In this situation (see, for instance, http://www.math.ucdavis.edu/~ralston/pub/Gaussnotes.pdf, part II) one can construct sequences \( \{\psi_n\} \) in \( \partial D \) satisfying \( \Delta_{\partial D} \psi_n + \tilde{k}^2_n \psi_n = O(k^{-1/2}) \) localized in a neighborhood of \( \gamma \) of radius \( O(k_n^{-1/2}) \). Using the *Ansatz* in (1) with \( \phi, \rho, a \) and \( b \) now independent of \( t \), one can adapt that.

\(^2\)There is a technical condition which selects which way to rotate, see J. Diff. Geom. 12(1977), 87-100.
construction to construct sequences \( \{u_n\} \) in \( D \) satisfying \( \Delta u_n + k_n^2 u_n = O(k_n^{-1/3}) \) and \( u_n = 0 \) on \( \partial D \). Since in this construction derivatives in the variable \( x \) are not determined by solving ODE’s along \( \gamma \), but simply by solving linear equations with periodic coefficients (namely multiples of \( \rho z^2 \)), the terms in (1) inherit the periods of the functions constructed for the quasi-modes for \( \Delta_{\partial D} \) when \( \alpha = 0 \). When \( \alpha = z_1 k_n^{-2/3} \) additional terms enter in the phase, and the leading order term comes from \( \phi_\alpha \) on \( \gamma \) as determined in (9). The quasi-mode eigenvalues \( k_n^2 \) are determined by the requirement that the function \( u \) in (1) is periodic – and hence well-defined when one goes around \( \gamma \). Letting \( L \) be the length of \( \gamma \) and \( y(t) \) be the parametrization used earlier, that condition becomes (for the fundamental quasi-mode)

\[
k_n L + \frac{z_1 k_n^{1/3}}{2} \int_0^L (\phi_y \cdot G_x \phi_y)^{2/3} ds - \frac{1}{2} (\theta_1 + \cdots + \theta_{d-1}) - \beta \pi = 2n\pi + O(k_n^{-1/3}),
\]

where \( \beta \) is zero or one, depending on the Morse index of \( \gamma \). In other words,

\[
k_n = \frac{2\pi n}{L} - \frac{z_1}{2^{1/3}} (\frac{2\pi n}{L})^{1/3} \bar{\kappa} + \frac{1}{2L} (\theta_1 + \cdots + \theta_{d-1}) + \beta \frac{\pi}{L} + O(n^{-1/3}), \tag{11}
\]

where we have used (10) to get

\[
\bar{\kappa} = \frac{1}{L} \int_0^L (\kappa(t))^{2/3} dt.
\]

The natural examples where explicit versions of these quasi-mode constructions could be done are solid tori and oblate spheroids: in each of those the outer equatorial geodesic satisfies the stability hypothesis. At least for the oblate spheroid it would be possible in principle to compare the quasi-modes with the exact eigenfunctions given as Lamé products (see H. Bateman, Partial Differential Equations, §8.52), but so far I have not found an account of that. The best comparison that I can offer is for the degenerate case of a disk of radius 1. Since the boundary is just the unit circle, there are no \( \theta \)'s in that case, and, using the value for \( z_1 = -2.33811 \) to six figures given earlier, \( L = 2\pi \), and \( \bar{\kappa} = 1 \), (11) becomes

\[
k_n = n + 1.85576n^{1/3} + O(n^{-1/3}).
\]

An exact eigenfunction in polar coordinates is \( J_n(\mu_n r) \cos(n\theta) \) where \( \mu_n \) is the first zero of \( J_n(r) \). Watson (A Treatise on the Theory of Bessel Functions, p. 521) has

\[
\mu_n = n + (1.855757)n^{1/3} + O(n^{-1/3})
\]