The Fundamental Theorems

The following formulations of the first and second fundamental theorems of calculus may suffice for any bounded functions that you actually *want* to integrate over bounded intervals.

1st Fundamental Theorem: Suppose that F is a continuous function on $[a,b]$ and that there is a partition, P, of $[a,b]$ such that $F'(x)$ exists and is continuous and bounded on $[a,b]\backslash P$. Then, if $g(x) = F'(x)$ on $[a,b]\backslash P$, g is Riemann-integrable on $[a,b]$, and

$$F(b) - F(a) = \int_a^b g(x)dx.$$

2nd Fundamental Theorem: Suppose that g is a function on $[a,b]$ such that g is bounded and continuous on $[a,b]\backslash P$ for some partition, P, of $[a,b]$. Then g is Riemann-integrable, and the set of continuous functions F on $[a,b]$ such that $F'(x) = g(x)$ on $[a,b]\backslash P$ is given by

$$F(x) = \int_a^x g(t)dt + C$$

as C ranges over all real numbers.

These theorems follow easily from what we have done in this Review and the following two results:

Theorem 1: Suppose that g is a function on $[a,b]$ such that g is bounded and continuous on $[a,b]\backslash P$ for some partition, P, of $[a,b]$. Then g is Riemann-integrable.

This is a special case of Theorem 6.10 in Rudin (pp. 126-7).

Theorem 2: Suppose that F is a continuous function on $[a,b]$. If there is a partition P of $[a,b]$ and a Riemann-integrable function g on $[a,b]$ such that $F'(x) = g(x)$ on $[a,b]\backslash P$, then

$$F(b) - F(a) = \int_a^b g(x)dx.$$

Proof. Given $\epsilon > 0$ choose a partition Q such that $U(g,Q) \leq L(g,Q) + \epsilon$. Then $U(g,P\cup Q) \leq L(g,P\cup Q) + \epsilon$ holds, since $P\cup Q$ is a refinement. Let $\{x_j : j = 0,\ldots,n\}$ be the points of $P\cup Q$. By the Mean Value Theorem there are points t_j^*, $x_{j-1} < t_j^* < x_j$, such that $F(x_j) - F(x_{j-1}) = F'(t_j^*)\Delta x_j$. Thus with the usual notation ($m_i = \inf\{g(x),\ x_{j-1} \leq x \leq x_j\}$ and $M_j = \sup\{g(x),\ x_{j-1} \leq x \leq x_j\}$)

$$m_j\Delta x_j \leq F(x_j) - F(x_{j-1}) \leq M_j\Delta x_j.$$

Summing those inequalities as j goes from 1 to n, one has

$$L(g,P\cup Q) \leq F(b) - F(a) \leq U(g,P\cup Q)$$

which, combined with $U(g, P \cup Q) \leq L(g, P \cup Q) + \epsilon$, gives

$$L(g, P \cup Q) \leq F(b) - F(a) \leq L(g, P \cup Q) + \epsilon \text{ and } L(g, P \cup Q) \leq \int_a^b g(x)dx \leq L(g, P \cup Q) + \epsilon.$$

Thus

$$|F(b) - F(a) - \int_a^b g(x)dx| \leq \epsilon$$

and the theorem follows since ϵ was an arbitrary positive number.