The shortest proof I know for $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y}) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$

I will assume that f, $\frac{\partial f}{\partial x}$, and $\frac{\partial f}{\partial y}$ are defined and continuous on a disk centered at (x, y) and that $\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})$ and $\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})$ are defined on that disk and continuous at (x, y).

Consider $f(x+h, y+h) - f(x+h, y) - f(x, y+h) + f(x, y)$. Let $g(x, y) = f(x+h, y) - f(x, y)$. Then you have

$$f(x+h, y+h) - f(x+h, y) - f(x, y+h) + f(x, y) = g(x, y+h) - g(x, y)$$

$$= h\frac{\partial g}{\partial y}(x, y+\theta h) \text{ for some } \theta,\ 0 < \theta < 1$$

$$= h[\frac{\partial f}{\partial y}(x+h, y+\theta h) - \frac{\partial f}{\partial y}(x, y+\theta h)]$$

$$= h^2\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(x+\beta h, y+\theta h) \text{ for some } \beta,\ 0 < \beta < 1,$$

where I used the Mean Value Theorem in the second variable in the second equality and the Mean Value Theorem in the first variable in the fourth equality.

However, if you let $r(x, y) = f(x, y+h) - f(x, y)$, you also have

$$f(x+h, y+h) - f(x+h, y) - f(x, y+h) + f(x, y) = r(x+h, y) - r(x, y).$$

Repeating the argument above, you will find θ' and β' between 0 and 1 such that

$$f(x+h, y+h) - f(x+h, y) - f(x, y+h) + f(x, y) = h^2\frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(x+\theta' h, y+\beta' h).$$

Now for each positive h which is small enough that all the points are in the disk you have found θ, θ', β and β' between 0 and 1 such that

$$\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(x+\beta h, y+\theta h) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(x+\theta' h, y+\beta' h).$$

Using the continuity of the mixed partial derivatives at (x, y), you can now let h go to zero to conclude

$$\frac{\partial}{\partial x}(\frac{\partial f}{\partial y})(x, y) = \frac{\partial}{\partial y}(\frac{\partial f}{\partial x})(x, y).$$