Dedekind and Decimals

One of the most convenient ways to define the real numbers is the method of "Dedekind Cuts". This assumes that you have defined the rational numbers and shown that they are an ordered field. A cut X is a set of rational numbers with the following properties:

a) X is nonempty and it is not equal to the whole set of rationals, $\mathbb{Q}$.

b) if $r \in X$ and $s < r$ then $s \in X$, and

c) if $r \in X$, then there is an $s \in X$ such that $s > r$.

You define $X \leq Y$ by $X \subset Y$, and $X + Y$ by

$$X + Y = \{r + s \ : r \in X, \ s \in Y\}.$$

Multiplication is a little trickier, since you do not want to include products of large negative rationals in your definition of XY. Taking care of that, and verifying all the axioms for an ordered field is a little tedious (see Rudin, Principles of Analysis, pp. 17-21). The best feature of Dedekind cuts as a definition is the simplicity of showing the existence of least upper and greatest lower bounds. If $\{X_\alpha\}_{\alpha\in A}$ is bounded above, then it is easy to show that $\cup_{\alpha\in A} X_\alpha$ is a cut, and that it is the least upper bound for $\{X_\alpha\}_{\alpha\in A}$. Likewise, if $\{X_\alpha\}_{\alpha\in A}$ is bounded below, then $\cap_{\alpha\in A} X_\alpha$ is a cut which is the greatest lower bound for $\{X_\alpha\}_{\alpha\in A}$.

However, most people – and all nonmathematicians – think of real numbers as decimal expansions. Rudin does not explain how to identify cuts with decimal expansions very well. [Rudin does not *like* decimal expansions.] So here is one way to do it.

Since $X \neq \mathbb{Q}$, there is a greatest integer, n_0, in X. Then there is a greatest integer n_1 (taken from the set $\{0, 1, \ldots, 9\}$) such that $n_0 + n_1/10 \in X$. Continuing this way, let n_k be the greatest integer such that $n_0 + n_1/10 + \cdots + n_k/10^k \in X$. Identify X with the decimal expansion $n_0.n_1n_2n_3\ldots$.

This assigns a decimal expansion to every cut. Could it assign the same decimal expansion to two different cuts? If $X \neq Y$, we can assume without loss of generality that $X < Y$, and let r be a rational number such that $r \in Y$ and $r \notin X$. Then, since Y is a cut, there is a rational number $s > r$ such that $s \in Y$ and, of course, $s \notin X$. Now to conclude that the decimal expansion for Y cannot be the same as the decimal expansion for X, you just need to show that there is a rational number t of the form

$$t = m_0 + m_1/10 + \cdots + m_N/10^N$$

such that $r < t < s$. This is a simple exercise.

This procedure assigns a *unique* decimal expansion to every real number. Some real numbers have more than one decimal expansion. Which expansion is it selecting?