In Metric Spaces Bolzano-Weierstrass Compactness is equivalent to Heine-Borel Compactness

(based on a homework problem by J.B. Garnett)

I will give the proof that B-W compactness implies H-B compactness here. The converse is an exercise.

Step 1: $\mathcal{X}$ is B-W compact implies that $\mathcal{X}$ is totally bounded.

Proof: Suppose not. Then there is an n_0 such that no finite set of open balls of radius $1/n_0$ covers $\mathcal{X}$. So, picking $x_1 \in \mathcal{X}$ at random, one can define a sequence in $\mathcal{X}$ by choosing

$$x_{m+1} \in (\cup_{i=1}^{m} B(x_i, 1/n_0))^c.$$

Since $d(x_{m+1}, x_i) \geq 1/n_0,\ i = 1, .., m$, this sequence has no convergent subsequences. (Why?)

The remainder of the argument is based on the following. For any nonempty $S \subset \mathcal{X}$ define $d(x, S) = \inf_{z \in S} d(x, z)$. Since for $z \in S$ we have $d(x, z) \leq d(x, y) + d(y, z)$, and hence $d(x, S) \leq d(x, y) + d(y, z)$ (Why?), it follows that

$$d(x, S) \leq d(x, y) + d(y, S). \tag{1}$$

We suppose that $\mathcal{X}$ is B-W compact and $\{\mathcal{O}_\alpha,\ \alpha \in A\}$ is an open cover of $\mathcal{X}$. Thus we need to show that $\{\mathcal{O}_\alpha,\ \alpha \in A\}$, contains a finite subcover.

Step 2: Let $\mathcal{O}_{\alpha,n} = \{x \in \mathcal{O}_\alpha : d(x, \mathcal{O}_\alpha^c) > 1/n\}$. If $\mathcal{X} \subset \cup_{\alpha \in A} \mathcal{O}_{\alpha,n_0}$ for some n_0, then $\{\mathcal{O}_\alpha,\ \alpha \in A\}$, contains a finite subcover.

Proof: By total boundedness there are $x_i,\ i = 1, .., N$, such that $\mathcal{X} \subset \cup_{i=1}^{N} B(x_i, 1/n_0)$, and by the hypothesis here $x_i \in \mathcal{O}_{\alpha_i,n_0}$ for some choice of $\alpha_i,\ i = 1, .., N$. By (1), if $y \in B(x_i, 1/n_0)$, then $d(y, \mathcal{O}_{\alpha_i}^c) \geq d(x_i, \mathcal{O}_{\alpha_i}^c) - d(y, x_i) > 0$. (Why?) Thus $B(x_i, 1/n_0) \subset \mathcal{O}_{\alpha_i}$ and $\mathcal{X} \subset \cup_{i=1}^{N} \mathcal{O}_{\alpha_i}$.

Step 3: If there is no n such that $\{\mathcal{O}_{\alpha,n},\ \alpha \in A\}$ covers $\mathcal{X}$, then we can choose a sequence $\{x_n\}$ by picking $x_n \in (\cup_{\alpha \in A} \mathcal{O}_{\alpha,n})^c$ for all n. If $x_{n_k} \to x_\infty$ as $k \to \infty$, then $x_\infty \in \mathcal{O}_{\alpha_0}$ for some α_0, and hence $B(x_\infty, 1/n_0) \subset \mathcal{O}_{\alpha_0}$ for some n_0, because $\mathcal{O}_{\alpha_0}$ is open. However, we chose $x_n \in \mathcal{O}_{\alpha_0,n}^c$ which implies $d(x_n, \mathcal{O}_{\alpha_0}^c) \leq 1/n$. Hence, using (1) again, we have

$$d(x_n, x_\infty) \geq d(x_\infty, \mathcal{O}_{\alpha_0}^c) - d(x_n, \mathcal{O}_{\alpha_0}^c) \geq \frac{1}{n_0} - \frac{1}{n}.$$

This contradicts the assumption that $x_{n_k} \to x_\infty$, and completes the proof.