Comments and Solutions for Assignment 4

1. Suppose that f is real-valued, bounded on [a,b] and f^3 is Riemann-integrable on [a,b]. Does it follow that f^2 is Riemann-integrable?

Yes, it does. Note that $\phi(y) = y^{2/3}$ is continuous on $\mathbb{R}$, so $f^2 = \phi(f^3)$ is Riemann-integrable because it a continuous function of a Riemann-integrable function.

2. Suppose that f is Riemann-integrable on $[c, 1]$ for every $c \in (0, 1]$. Define the improper Riemann integral of f on [0,1] by

$$\int_0^1 f(x)dx = \lim_{c \downarrow 0} \int_c^1 f(x)dx \tag{1}$$

if this limit exists and is finite.

a) Show that this agrees with the Riemann integral when f is Riemann-integrable on [0,1].

b) Construct an f for which the limit in (2) exists which is not Riemann integrable, and for which the limit in (2) does not exist when f is replaced by $|f|$.

For a) you can argue that when f is Riemann-inegrable

$$\int_0^1 f(x)dx = \int_0^c f(x)dx + \int_c^1 f(x)dx.$$

So

$$|\int_0^1 f(x)dx - \int_c^1 f(x)dx| \leq c \sup_{x \in [0,1]} |f(x)|$$

[remember that to be Riemann-integrable you first have to be bounded].

For b) you need the derivative of a function which is continuous on $[0, 1]$ but very wiggly near $x = 0$. Take $F(x) = x \sin(1/x)$ and $f(x) = F'(x) = \sin(1/x) - (1/x)\cos(1/x)$. Then, by the fundamental theorem $\int_c^1 F'(x)dx = \sin(1) - c\sin(1/c)$ which converges to $\sin(1)$ as c goes to zero. However,

$$|f(x)| \geq \frac{1}{x} |\cos(\frac{1}{x})| - 1.$$

This is quite large when x is small and, for instance, $\cos(1/x) \geq 1/\sqrt{2}$. That happens on the intervals $I_n = [(n\pi + \pi/4)^{-1}, (n\pi - \pi/4)^{-1}]$. So on I_n we have

$$|f(x)| \geq (n - 1/4)\pi 2^{-1/2} - 1 =_{\text{def}} a_n.$$

Now let $x_N = 1/(N\pi + \pi/4)$, and consider the partition, P_N, of $[x_N, 1]$ given by 1 and the endpoints of the I_n's for $n \leq N$. Then one has

$$\int_{x_N}^1 |f(x)|dx \geq L(|f|, P_N) \geq \sum_{n=1}^N a_n l(I_n) = \sum_{n=1}^N (\frac{a_n}{2\pi}) \frac{1}{n^2 - 1/16}.$$

However, the final sum above diverges by by limit comparison with $\sum n^{-1}$, and

$$\lim_{N\to\infty}\int_{x_N}|f(x)|dx=\infty.$$

3. (# 4 on Basic F'07, also in a more obscure version # 3 W'06) Suppose that $f:\mathbb{R}\to\mathbb{R}$ is twice differentiable and $|f''(x)|\le B$ for all x.

a) Prove that

$$|2Af(0)-\int_{-A}^{A}f(x)dx|\le 2BA^3/3$$

b) Use the result of part a) to justify the estimate

$$|\int_a^b f(x)dx-\frac{b-a}{n}\sum_{k=1}^{n}f(a+\frac{2k-1}{2n}(b-a))|\le\frac{C}{n^2}$$

where C does not depend on n.

For a): The Taylor series expansion of f about x=0 gives

$$f(x)=f(0)+f'(0)x+\frac{x^2}{2}f''(x^*)$$

for some x^* between x and 0. So

$$|\int_{-A}^{A}f(x)dx-2Af(0)|\le B\int_{-A}^{A}x^2dx=B\frac{2A^3}{3}.$$

For b): Just apply a) with $x=0$ replaced by $x=a+\frac{2k-1}{2n}(b-a))$ and $A=(b-a)/(2n)$. Then the estimate from a) becomes

$$|\int_{(k-1)(b-a)/n}^{k(b-a)/n}f(x)dx-\frac{(b-a)}{n}f(a+\frac{2k-1}{2n}(b-a))|\le B\frac{(b-a)^3}{3n^3}.$$

Adding those up as k goes from 1 to n gives the result with $C=B(b-a)^3/3$.

4. Suppose that f on [0,1] is defined by

$$f(x)=\begin{cases}0, & \text{if } x \text{ is irrational}\\ 1/m, & \text{if x=n/m is lowest terms.}\end{cases}$$

Show that f is Riemann-integrable on [0,1] and its integral is 0.

Recall that this function is continuous at all irrational numbers, and hence has only countably many discontinuities – so it has to be Riemann-integrable. To prove that without appealing to the general characterization of Riemann-integrable functions you can argue as follows. Given $\epsilon>0$, let $r_1,...,r_N$ be the rational numbers in $[0,1]$ with denominators less that $2\epsilon^{-1}$ Take a partition containing the points $r_i\pm\delta$ where $\delta>0$ is small enough that the intervals $[r_j-\delta,r_j+\delta]$, $j=$

$1, .., N$ are disjoint and $2N\delta < \epsilon/2$. Consider the upper sum $U(f, P)$ for this partition. Since $0 \leq f(x) \leq 1$, the total contribution to $U(f, P)$ from the intervals $[r_j - \delta, r_j + \delta]$, $j = 1, .., N$ is bounded by $(2\delta)N < \epsilon/2$, and, the contribution from intervals not containing r_j's is at most $\epsilon/2$, since $f(x) < \epsilon/2$ on those intervals. So $U(f, P) < \epsilon$. Since $L(f, P) \geq 0$, we have

$$0 \leq L(f,P) \leq \int_{\underline{0}}^{1} f(x)dx \leq \int_{o}^{\overline{1}} f(x)dx \leq U(f,P) < \epsilon$$

which, since ϵ is arbitrary, implies that f is Riemann-integrable with integral 0.

5. Build a sequence of closed subintervals of $[0, 1]$ as follows: set $I_1 = [0, 1]$, $I_2 = [0, 1/2]$, $I_3 = [1/2, 1]$, $I_4 = [0, 1/4]$, $I_5 = [1/4, 1/2]$, and so on, filling out [0,1] over and over, and halving the length of the intervals each time you start over. Let $f_n(x) = 1$ on I_n and 0 elsewhere. Show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$ even though $\lim_{n\to\infty} f_n(x)$ fails to exist for any $x \in [0, 1]$.

This is a useful example, but it's a trivial problem. Note $0 \leq f_n(x) \leq 1$ for $x \in [0, 1]$ and $f_n(x) = 0$ outside an interval whose length goes to zero as $n \to \infty$. So $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$. On the other hand, for each x, $f_n(x) = 1$ for infinitely many n and $f_n(x) = 0$ for infinitely many n.

6. (Rudin) Letting $\{x\}$ denote the fractional part of x, i.e. $\{x\}$ is the largest integer less than or equal x, consider $f(x) = \sum_{n=1}^{\infty} \{nx\}/n^2$. Find the discontinuities of f and show that they form a countable dense set in $\mathbb{R}$. Show that nevertheless f is Riemann-integrable on every bounded interval in $\mathbb{R}$.

The function $g_n(x) = \{nx\}/n^2$ is continuous except at the points $x = k/n$, $k \in \mathbb{Z}$ ($\mathbb{Z}$ stands for the integers). Moreover at each of those points it "jumps down", i.e. $\lim_{x\uparrow k/n} g_n(x) = \lim_{x\downarrow k/n} g_n(x) + 1/n^2$. So, when you sum up the g_n's, the jumps do not cancel each other out. One should probably be a little more rigorous here. The main observation is that the sum defining $f(x)$ converges uniformly: $|S_N(x) - S_M(x)| \leq \sum_{n=M+1}^{\infty} 1/n^2$ for $x \in \mathbb{R}$, where S_N is the usual partial sum. Since $S_N(x)$ converges uniformly to $f(x)$,

$$\lim_{x\downarrow x_0} f(x) = \lim_{N\to\infty} \lim_{x\downarrow x_0} S_N(x) \text{ and } \lim_{x\uparrow x_0} f(x) = \lim_{N\to\infty} \lim_{x\uparrow x_0} S_N(x),$$

provided that $\lim_{x\downarrow x_0} S_N(x)$ and $\lim_{x\uparrow x_0} S_N(x)$ exist for each N. This shows that $f(x)$ is continuous at the irrationals, since the g_n's are, and discontinuous at the rationals.

However, on any bounded interval one sees that $S_N(x) = \sum_{n=1}^{N} g_n(x)$ is a function with only finitely many points of discontinuity for each N, and therefore is Riemann integrable. Thus the sequence of Riemann-integrable functions $\{S_N(x)\}$ converges uniformly to $f(x)$, and we conclude that f is Riemann-integrable.