1. Suppose that \( f \) is real-valued, bounded on \([a,b]\) and \( f^3 \) is Riemann-integrable on \([a,b]\). Does it follow that \( f^2 \) is Riemann-integrable?

2. Suppose that \( f \) is Riemann-integrable on \([c,1]\) for every \( c \in (0,1] \). Define the improper Riemann integral of \( f \) on \([0,1]\) by

\[
\int_0^1 f(x) \, dx = \lim_{c \to 0} \int_c^1 f(x) \, dx
\]

if this limit exists and is finite.

(a) Show that this agrees with the Riemann integral when \( f \) is Riemann-integrable on \([0,1]\).

(b) Construct an \( f \) for which the limit in (2) exists which is not Riemann integrable, and for which the limit in (2) does not exist when \( f \) is replaced by \(|f|\).

3. (# 4 on Basic F’07, also in a more obscure version # 3 W’06) Suppose that \( f : \mathbb{R} \to \mathbb{R} \) is twice differentiable and \(|f''(x)| \leq B\) for all \( x \).

(a) Prove that

\[
|2Af(0) - \int_{-A}^A f(x) \, dx| \leq 2BA^3/3
\]

(b) Use the result of part a) to justify the estimate

\[
\left| \int_a^b f(x) \, dx - \frac{b-a}{n} \sum_{k=1}^{n} f\left(a + \frac{2k-1}{2n}(b-a)\right) \right| \leq \frac{C}{n^2}
\]

where \( C \) does not depend on \( n \).

4. Suppose that \( f \) on \([0,1]\) is defined by

\[
f(x) = \begin{cases} 
0, & \text{if } x \text{ is irrational} \\
1/m, & \text{if } x=n/m \text{ is lowest terms}.
\end{cases}
\]

Show that \( f \) is Riemann-integrable on \([0,1]\) and its integral is 0.

5. Build a sequence of closed subintervals of \([0,1]\) as follows: set \( I_1 = [0,1] \), \( I_2 = [0,1/2] \), \( I_3 = [1/2,1] \), \( I_4 = [0,1/4] \), \( I_5 = [1/4,1/2] \), and so on, filling out \([0,1]\) over and over, and halving the length of the intervals each time you start over. Let \( f_n(x) = 1 \) on \( I_n \) and 0 elsewhere. Show that \( \lim_{n \to \infty} \int_0^1 f_n(x) \, dx = 0 \) even though \( \lim_{n \to \infty} f_n(x) \) fails to exist for any \( x \in [0,1] \).

6. (Rudin) Letting \( \{x\} \) denote the fractional part of \( x \), i.e. \( x \) minus the largest integer less than or equal to \( x \) consider \( f(x) = \sum_{n=1}^{\infty} \{nx\}/n^2 \). Find the discontinuities of \( f \) and show that they form a countable dense set in \( \mathbb{R} \). Show that \( f \) is nevertheless Riemann-integrable on every bounded interval.