Assignment 4 – due Tuesday, September 8

1. Suppose that f is real-valued, bounded on [a,b] and f^3 is Riemann-integrable on [a,b]. Does it follow that f^2 is Riemann-integrable?

2. Suppose that f is Riemann-integrable on $[c, 1]$ for every $c \in (0, 1]$. Define the improper Riemann integral of f on [0,1] by

$$\int_0^1 f(x)dx = \lim_{c \downarrow 0} \int_c^1 f(x)dx \tag{1}$$

if this limit exists and is finite.

a) Show that this agrees with the Riemann integral when f is Riemann-integrable on [0,1].

b) Construct an f for which the limit in (2) exists which is not Riemann integrable, and for which the limit in (2) does not exist when f is replaced by $|f|$.

3. (# 4 on Basic F'07, also in a more obscure version # 3 W'06) Suppose that $f : \mathbb{R} \to \mathbb{R}$ is twice differentiable and $|f''(x)| \leq B$ for all x.

a) Prove that

$$|2Af(0) - \int_{-A}^{A} f(x)dx| \leq 2BA^3/3$$

b) Use the result of part a) to justify the estimate

$$|\int_a^b f(x)dx - \frac{b-a}{n} \sum_{k=1}^{n} f(a + \frac{2k-1}{2n}(b-a))| \leq \frac{C}{n^2}$$

where C does not depend on n.

4. Suppose that f on [0,1] is defined by

$$f(x) = \begin{cases} 0, & \text{if } x \text{ is irrational} \\ 1/m, & \text{if x=n/m is lowest terms.} \end{cases}$$

Show that f is Riemann-integrable on [0,1] and its integral is 0.

5. Build a sequence of closed subintervals of $[0, 1]$ as follows: set $I_1 = [0, 1]$, $I_2 = [0, 1/2]$, $I_3 = [1/2, 1]$, $I_4 = [0, 1/4]$, $I_5 = [1/4, 1/2]$, and so on, filling out [0,1] over and over, and halving the length of the intervals each time you start over. Let $f_n(x) = 1$ on I_n and 0 elsewhere. Show that $\lim_{n\to\infty} \int_0^1 f_n(x)dx = 0$ even though $\lim_{n\to\infty} f_n(x)$ fails to exist for any $x \in [0, 1]$.

6. (Rudin) Letting $\{x\}$ denote the fractional part of x, i.e. x minus the largest integer less than or equal to x consider $f(x) = \sum_{n=1}^{\infty} \{nx\}/n^2$. Find the discontinuities of f and show that they form a countable dense set in $\mathbb{R}$. Show that f is nevertheless Riemann-integrable on every bounded interval.