Comments and Solutions for Assignment 3

1. (Basic S’09) Set \( a_0 = 0 \) and define a sequence \( \{a_n\}_{n=1}^\infty \) via the recurrence

\[
a_{n+1} = \sqrt{6 + a_n} \quad \text{for all } n \geq 1.
\]

Show that this sequence converges and determine its limit.

This is a standard problem, and you can see easily that the limit must be 3. However, the problem asks you to prove that carefully. Note first that, if \( a_n < 3 \) then \( a_{n+1} < 3 \) (because \( \sqrt{6 + 3} = 3 \) and \( f(x) = \sqrt{6 + x} \) is increasing). So \( 0 < a_n < 3 \) for \( n \geq 1 \). I claim that \( a_n < a_{n+1} \) for all \( n \). The easiest way to show that may be the argument that one of you suggested in class: induction. The key ingredient is just the fact mentioned above that \( f(x) = \sqrt{6 + x} \) is strictly increasing. Since \( a_0 = 0 < \sqrt{6} = a_1 \), the induction hypothesis is \( a_k < a_{k+1} \) for \( k < n \). Then you have

\[
a_{n+1} = \sqrt{6 + a_n} > \sqrt{6 + a_{n-1}} = a_n,
\]

where the central inequality follows from the induction hypothesis plus \( \sqrt{6 + x} \) is strictly increasing. Thus the sequence \( \{a_n\} \) is monotone increasing and bounded above. So it has a limit \( L \). Since \( f(x) = \sqrt{6 + x} \) is continuous on \([0, \infty)\), we have

\[
L = \lim_{n \to \infty} a_{n+1} = \lim_{n \to \infty} \sqrt{6 + a_n} = \sqrt{6 + L},
\]

and \( L = 3 \).

This argument works in a pretty general setting. Suppose that \( f(x) \) is any strictly increasing continuous function defined on a closed interval \( I \subset \mathbb{R} \), and define a sequence by \( a_n = f(a_{n-1}) \), \( n = 1, 2, \ldots \) – assume that \( a_n \in I \) for all \( n \) so that the sequence is defined. If \( f(a_0) = a_0 \), then \( a_n = a_0 = 0 \) for all \( n \). If \( f(a_0) \neq a_0 \), then \( a_1 < a_0 \) or \( a_1 > a_0 \). In the first case set \( a_\infty = \sup S(a_0) \), where \( S(a_0) = \{ x \in I : f(x) = x \quad \text{and} \quad x < a_0 \} \). If \( S(a_0) = \emptyset \), I claim \( \lim_{n \to \infty} a_n = -\infty \), but assume that it is not empty for now. Note that \( f(a_\infty) = a_\infty \) because \( f \) is continuous. Since \( f(x) \) is increasing and \( f(a_\infty) = a_\infty \), it follows that \( a_\infty < a_n \) for all \( n \). Since \( a_{n+1} < a_n \) by exactly the induction argument above, \( \lim_{n \to \infty} a_n = L \geq a_\infty \) and \( f(L) = L \). If \( L > a_\infty \), we have a contradiction to the definition of \( a_\infty \). So \( L = a_\infty \). Even if \( S(a_0) = \emptyset \), the sequence is monotone decreasing. If it had a limit, then the limit would be in \( S(a_0) \). So it does not have a limit and must be unbounded below.

The argument in the second case is the same except \( a_\infty = \inf \{ x : f(x) = x \quad \text{and} \quad x > a_0 \} \).

2. Determine whether \( \sum a_n \) converges for the following choices of \( a_n \).

a) \( a_n = (n^{1/n} - 1)^n \)

b) \( a_n = (1 + z^n)^{-1} \). The answer depends on the complex number \( z \).

Series a) converges. You can use the root test: \( a_n^{1/n} = n^{1/n} - 1 \) which goes to zero as \( n \to \infty \). To see that, starting from \( n^{1/n} \geq 1 \) and avoiding l’Hôpital’s rule, let \( x = n^{1/n} - 1 \). Then by the binomial theorem

\[
n = (1 + x)^n = 1 + nx + n(n - 1)x^2/2 + \cdots + x^n \geq n(n - 1)x^2/2.
\]
So $0 \leq x \leq 2/\sqrt{n-1}$. That use of the binomial theorem came from Rudin.

The series b) converges for $|z| > 1$ and diverges for $|z| \leq 1$. If $|z| \leq 1$, then

$$|a_n| \geq \frac{1}{1 + |z|^n} \geq \frac{1}{1 + |z|} \geq \frac{1}{2}.$$ 

So the terms do not go to zero, and the series diverges. Of course, if $z^n = -1$ for some $n$, the series isn’t even defined.

If $|z| > 1$, then

$$\frac{1}{1 + z^n} < \frac{1}{|z|^n - 1} = \frac{1}{|z|^n}(\frac{1}{1 - |z|^{-n}}) < \frac{2}{|z|^n}$$

for $n$ sufficiently large. So the series converges by comparison with the geometric series.

3. Suppose that $a_n \geq 0$ and $\sum a_n = \infty$, i.e. the partial sums are unbounded. Show that

$$\sum \frac{a_n}{1 + a_n} = \infty.$$ 

What about

$$\sum \frac{a_n}{1 + n^2 a_n} \text{ and } \sum \frac{a_n}{1 + n a_n}?$$

The first question is pretty easy: if there is a subsequence $\{a_{n_k}\}$ such that $\lim_{k \to \infty} a_{n_k} = \infty$, then $\lim_{k \to \infty} \frac{a_{n_k}}{1 + a_{n_k}} = 1$ and the series diverges because the terms do not go to zero. If there is no subsequence going to infinity, there is a $B$ such that $a_n \leq B$ for all $n$, and $a_n/(1 + a_n) \geq a_n/(1 + B)$. Then the series diverges by comparison with $\sum a_n/(1 + B)$.

Since $a_n/(1 + n^2 a_n) < 1/n^2$ for all $n$, the series $\sum a_n/(1 + n^2 a_n)$ always converges by comparison with $\sum 1/n^2$.

The interesting case is the last one which can converge or diverge: if $a_n = 1$ for all $n$, it becomes $\sum 1/(n + 1)$ which diverges. However, if $a_{2^m} = 1$ and $a_n = 0$ for $n \neq 2^m$, then

$$\sum_{n=1}^{\infty} \frac{a_n}{1 + n a_n} = \sum_{m=0}^{\infty} \frac{1}{1 + 2^m}$$

which converges.

4. Suppose that $\sum a_n$ converges, $\{b_n\}$ is monotonic increasing and bounded above. Show that $\sum a_n b_n$ converges. [This is not as easy as it might sound remember that these are not necessarily series with nonnegative terms.]

This one works by summation by parts: with $s_n = \sum_{k=1}^{n} a_k$ the summation by parts formula gives

$$\sum_{n=M+1}^{N} a_n b_n = \sum_{n=M+1}^{N-1} s_n (b_n - b_{n+1}) + s_{N} b_{N} - s_{M} b_{M+1}. $$
Note that both \( \{s_n\} \) and \( \{b_n\} \) are convergent sequences: by assumption \( \lim_{n \to \infty} s_n \) is \( \sum a_n \), and \( \lim_{n \to \infty} b_n = \sup \{b_n\} \). So \( \{s_n\} \) must be bounded, and we have \( |s_n| \leq C \) for all \( n \). Also, given \( \epsilon > 0 \), there is an \( R \) such that \( |s_N b_N - s_M b_{M+1}| < \epsilon \) when \( N, M \geq R \). So for \( N > M \geq R \) we have

\[
|\sum_{n=1}^{N} a_n b_n - \sum_{n=1}^{M} a_n b_n| \leq C \sum_{M+1}^{N-1} (b_n - b_{n+1}) + \epsilon = C(b_N - b_{M+1}) + \epsilon.
\]

Taking \( R \) larger if necessary, we can assume \( C(b_N - b_{M+1}) < \epsilon \) for \( N, M \geq R \). Thus the partial sums of \( \sum a_n b_n \) form a Cauchy sequence, and this series converges.

5. (Basic S’07) Suppose that \( A \) is a symmetric, real, \( n \times n \) matrix with eigenvalues \( \lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n \). Find the sets

\[
X = \{ x \in \mathbb{R}^n : \lim_{k \to \infty} (x^t A^{2k} x)^{1/k} \text{ exists } \}, \text{ and } \]

\[
L = \{ \lim_{k \to \infty} (x^t A^{2k} x)^{1/k} : x \in X \}.
\]

You get to assume everything that you know about linear algebra here.

So you can assume without proof that there is a real, orthogonal matrix such that \( O^t A O \) is diagonal. Then, for any \( x \in \mathbb{R}^n \),

\[
(x^t A^{2k} x)^{1/k} = (c_1^2 \lambda_1^{2k} + c_2^2 \lambda_2^{2k} + \cdots + c_n^2 \lambda_n^{2k})^{1/k},
\]

where \( x = \sum_{m=1}^{n} c_m v_m \) and \( \{v_m\} \) is an orthonormal basis consisting of eigenvectors of \( A \). I claim that \( X = \mathbb{R}^n \). To describe the limit define \( P_\lambda x \) to be the orthogonal projection of \( x \) onto the eigenspace belonging to the eigenvalue \( \lambda \). Note that, since we have not assumed that the eigenvalues of \( A \) are distinct, that space may have dimension greater than one. Then, for \( x = 0 \) the limit is 0 and for \( x \neq 0 \) the limit is \( \lambda_2 \), where \( \lambda_0 \) is the largest eigenvalue such that \( P_{\lambda_0} x \neq 0 \). Modulo the linear algebra, proving that is easy. Let \( \lambda_0 \) be the largest eigenvalue such that \( P_{\lambda_0} x \neq 0 \), and suppose that in our labeling, \( \lambda_0 = \lambda_{m_0} = \cdots \lambda_{m_0+k} \) and \( \lambda_m < \lambda_0 \) for \( m < m_0 \).

Then

\[
(x^t A^{2k} x)^{1/k} = \lambda_0^2 |P_{\lambda_0} x|^{2/k} ((c_1^2 (\lambda_1/\lambda_0)^{2k} + \cdots + c_{m_0-1}^2 (\lambda_{m_0-1}/\lambda_0)^{2k}) |P_{\lambda_0} x|^{-2} + 1)^{1/k}
\]

Note that

\[
((c_1^2 (\lambda_1/\lambda_0)^{2k} + \cdots + c_{m_0-1}^2 (\lambda_{m_0-1}/\lambda_0)^{2k}) |P_{\lambda_0} x|^{-2} + 1)^{1/k} = (1 + M(k))^{1/k},
\]

where \( M(k) \) goes to zero as \( k \to \infty \). Since \( x^{1/k} \) converges to 1 uniformly as \( k \to \infty \) on compact subintervals of \( (0, \infty) \) – by a simpler version of the argument in problem 2.a) – we have \( \lim_{k \to \infty} (x^t A^{2k} x)^{1/k} = \lambda_0^2 \).

6. Show that the series

\[
\sum_{n=1}^{\infty} \frac{\exp(inx)}{n}
\]
converges uniformly on any compact subinterval of $(0, 2\pi)$.

Note that $s_N = \sum_{n=0}^{N} e^{inx} = \frac{1-e^{i(N+1)x}}{1-e^{ix}}$, and so for $x$ any compact interval $I \subset (0, 2\pi)$ there is a $B$ such that $|s_N| \leq B$ for all $N$ and all $x \in I$. Now, as in problem 4, summation by parts gives

$$\sum_{n=M+1}^{N} \frac{e^{inx}}{n} = \sum_{n=M+1}^{N-1} s_n \left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{s_N}{N} - \frac{s_M}{M}.$$ 

Thus for $x \in I$ and $N > M$

$$|\sum_{n=1}^{N} \frac{e^{inx}}{n} - \sum_{n=1}^{M} \frac{e^{inx}}{n}| \leq B \sum_{n=M+1}^{N-1} \left(\frac{1}{n} - \frac{1}{n+1}\right) + B \left(\frac{1}{N} + \frac{1}{M}\right) = B (\frac{1}{M+1} + \frac{1}{M}).$$

From this one sees easily that $S_N(x) = \sum_{n=1}^{N} e^{inx}/n$ is a sequence of functions that converges uniformly on $I$. 