Comments and Solutions for Assignment 3

1. (Basic S'09) Set $a_0 = 0$ and define a sequence $\{a_n\}_{n=1}^{\infty}$ via the recurrence

$$a_{n+1} = \sqrt{6 + a_n} \qquad \text{for all } n \geq 1.$$

Show that this sequence converges and determine its limit.

This is a standard problem, and you can see easily that the limit must be 3. However, the problem asks you to prove that carefully. Note first that, if $a_n < 3$ then $a_{n+1} < 3$ (because $\sqrt{6+3} = 3$ and $f(x) = \sqrt{6+x}$ is increasing). So $0 < a_n < 3$ for $n \geq 1$. I claim that $a_n < a_{n+1}$ for all n. The easiest way to show that may be the argument that one of you suggested in class: induction. The key ingredient is just the fact mentioned above that $f(x) = \sqrt{6+x}$ is strictly increasing. Since $a_0 = 0 < \sqrt{6} = a_1$, the induction hypothesis is $a_k < a_{k+1}$ for $k < n$. Then you have

$$a_{n+1} = \sqrt{6 + a_n} > \sqrt{6 + a_{n-1}} = a_n,$$

where the central inequality follows from the induction hypothesis plus $\sqrt{6+x}$ is strictly increasing. Thus the sequence $\{a_n\}$ is monotone increasing and bounded above. So it has a limit L. Since $f(x) = \sqrt{6+x}$ is continuous on $[0, \infty)$, we have

$$L = \lim_{n\to\infty} a_{n+1} = \lim_{n\to\infty} \sqrt{6 + a_n} = \sqrt{6 + L},$$

and $L = 3$.

This argument works in a pretty general setting. Suppose that $f(x)$ is *any* strictly increasing continuous function defined on a closed interval $I \subset \mathbb{R}$, and define a sequence by $a_n = f(a_{n-1})$, $n = 1, 2, \ldots$ – assume that $a_n \in I$ for all n so that the sequence is defined. If $f(a_0) = a_0$, then $a_n = a_0$ for all n. If $f(a_0) \neq a_0$, then $a_1 < a_0$ or $a_1 > a_0$. In the first case set $a_\infty = \sup S(a_0)$, where $S(a_0) = \{x \in I : f(x) = x \text{ and } x < a_0\}$. If $S(a_0) = \emptyset$, I claim $\lim_{n\to\infty} a_n = -\infty$, but assume that it is not empty for now. Note that $f(a_\infty) = a_\infty$ because f is continuous. Since $f(x)$ is increasing and $f(a_\infty) = a_\infty$, it follows that $a_\infty < a_n$ for all n. Since $a_{n+1} < a_n$ by exactly the induction argument above, $\lim_{n\to\infty} a_n = L \geq a_\infty$ and $f(L) = L$. If $L > a_\infty$, we have a contradiction to the definition of a_∞. So $L = a_\infty$. Even if $S(a_0) = \emptyset$, the sequence is monotone decreasing. If it had a limit, then the limit would be in $S(a_0)$. So it does not have a limit and must be unbounded below.

The argument in the second case is the same except $a_\infty = \inf\{x : f(x) = x \text{ and } x > a_0\}$.

2. Determine whether $\sum a_n$ converges for the following choices of a_n.
   a) $a_n = (n^{1/n} - 1)^n$
   b) $a_n = (1 + z^n)^{-1}$. The answer depends on the complex number z.

Series a) converges. You can use the root test: $a_n^{1/n} = n^{1/n} - 1$ which goes to zero as $n \to \infty$. To see that, starting from $n^{1/n} \geq 1$ and avoiding l'Hôpital's rule, let $x = n^{1/n} - 1$. Then by the binomial theorem

$$n = (1+x)^n = 1 + nx + n(n-1)x^2/2 + \cdots + x^n \geq n(n-1)x^2/2.$$

So $0 \le x \le 2/\sqrt{n-1}$. That use of the binomial theorem came from Rudin.

The series b) converges for $|z| > 1$ and diverges for $|z| \le 1$. If $|z| \le 1$, then

$$|a_n| \ge \frac{1}{1+|z|^n} \ge \frac{1}{1+|z|} \ge \frac{1}{2}.$$

So the terms do not go to zero, and the series diverges. Of course, if $z^n = -1$ for some n, the series isn't even defined.

If $|z| > 1$, then

$$|\frac{1}{1+z^n}| < \frac{1}{|z|^n - 1} = \frac{1}{|z|^n}(\frac{1}{1-|z|^{-n}}) < \frac{2}{|z|^n}$$

for n sufficiently large. So the series converges by comparison with the geometric series.

3. Suppose that $a_n \ge 0$ and $\sum a_n = \infty$, i.e. the partial sums are unbounded. Show that

$$\sum \frac{a_n}{1+a_n} = \infty.$$

What about

$$\sum \frac{a_n}{1+n^2 a_n} \text{ and } \sum \frac{a_n}{1+n a_n}?$$

The first question is pretty easy: if there is a subsequence $\{a_{n_k}\}$ such that $\lim_{k\to\infty} a_{n_k} = \infty$, then $\lim_{k\to\infty} \frac{a_{n_k}}{1+a_{n_k}} = 1$ and the series diverges because the terms to do not go to zero. If there is no subsequence going to infinity, there is a B such that $a_n \le B$ for all n, and $a_n/(1+a_n) \ge a_n/(1+B)$. Then the series diverges by comparison with $\sum a_n/(1+B)$.

Since $a_n/(1+n^2 a_n) < 1/n^2$ for all n, the series $\sum a_n/(1+n^2 a_n)$ always converges by comparison with $\sum 1/n^2$.

The interesting case is the last one which can converge or diverge: if $a_n = 1$ for all n, it becomes $\sum 1/(n+1)$ which diverges. However, if $a_{2^m} = 1$ and $a_n = 0$ for $n \ne 2^m$, then

$$\sum_{n=1}^{\infty} \frac{a_n}{1+n a_n} = \sum_{m=0}^{\infty} \frac{1}{1+2^m}$$

which converges.

4. Suppose that $\sum a_n$ converges, $\{b_n\}$ is monotonic increasing and bounded above. Show that $\sum a_n b_n$ converges. [This is not as easy as it might sound remember that these are not necessarily series with nonnegative terms.]

This one works by summation by parts: with $s_n = \sum_{k=1}^{n} a_k$ the summation by parts formula gives

$$\sum_{n=M+1}^{N} a_n b_n = \sum_{n=M+1}^{N-1} s_n(b_n - b_{n+1}) + s_N b_N - s_M b_{M+1}.$$

Note that both $\{s_n\}$ and $\{b_n\}$ are convergent sequences: by assumption $\lim_{n\to\infty} s_n$ is $\sum a_n$, and $\lim_{n\to\infty} b_n = \sup\{b_n\}$. So $\{s_n\}$ must be bounded, and we have $|s_n| \le C$ for all n. Also, given $\epsilon > 0$, there is an R such that $|s_N b_N - s_M b_{M+1}| < \epsilon$ when $N, M \ge R$. So for $N > M \ge R$ we have

$$|\sum_{n=1}^{N} a_n b_n - \sum_{n=1}^{M} a_n b_n| \le C \sum_{M+1}^{N-1} (b_n - b_{n+1}) + \epsilon = C(b_N - b_{M+1}) + \epsilon.$$

Taking R larger if necessary, we can assume $C(b_N - b_{M+1}) < \epsilon$ for $N, M \ge R$. Thus the partial sums of $\sum a_n b_n$ form a Cauchy sequence, and this series converges.

5. (Basic S'07) Suppose that A is a symmetric, real, $n \times n$ matrix with eigenvalues $\lambda_1 \le \lambda_2 \le \cdots \le \lambda_n$. Find the sets

$$X = \{x \in \mathbb{R}^n : \lim_{k\to\infty} (x^t A^{2k} x)^{1/k} \text{ exists } \}, \text{ and}$$

$$L = \{\lim_{k\to\infty} (x^t A^{2k} x)^{1/k} : x \in X\}.$$

You get to assume everything that you know about linear algebra here.

So you can assume without proof that there is a real, orthogonal matrix such that $O^t A O$ is diagonal. Then, for any $x \in \mathbb{R}^n$,

$$(x^t A^{2k} x)^{1/k} = (c_1^2 \lambda_1^{2k} + c_2^2 \lambda_2^{2k} + \cdots + c_n^2 \lambda_n^{2k})^{1/k},$$

where $x = \sum_{m=1}^{n} c_m v_m$ and $\{v_m\}$ is an orthonormal basis consisting of eigenvectors of A. I claim that $X = \mathbb{R}^n$. To describe the limit define $P_\lambda x$ to be the orthogonal projection of x onto the eigenspace belonging to the eigenvalue λ. Note that, since we have not assumed that the eigenvalues of A are distinct, that space may have dimension greater than one. Then, for $x = 0$ the limit is 0 and for $x \ne 0$ the limit is λ_0^2, where λ_0 is the largest eigenvalue such that $P_{\lambda_0} x \ne 0$. Modulo the linear algebra, proving that is easy. Let λ_0 be the largest eigenvalue such that $P_{\lambda_0} x \ne 0$, and suppose that in our labeling, $\lambda_0 = \lambda_{m_0} = \cdots \lambda_{m_0+k}$ and $\lambda_m < \lambda_0$ for $m < m_0$. Then

$$(x^t A^{2k} x)^{1/k} = \lambda_0^2 |P_{\lambda_0} x|^{2/k} ((c_1^2 (\lambda_1/\lambda_0)^{2k} + \cdots + c_{m_0-1}^2 (\lambda_{m_0-1}/\lambda_0)^{2k}) |P_{\lambda_0} x|^{-2} + 1)^{1/k}$$

Note that

$$((c_1^2 (\lambda_1/\lambda_0)^{2k} + \cdots + c_{m_0-1}^2 ((\lambda_{m_0-1}/\lambda_0)^{2k}) |P_{\lambda_0} x|^{-2} + 1)^{1/k} = (1 + M(k))^{1/k},$$

where $M(k)$ goes to zero as $k \to \infty$. Since $x^{1/k}$ converges to 1 uniformly as $k \to \infty$ on compact subintervals of $(0, \infty)$ – by a simpler version of the argument in problem 2.a) – we have $\lim_{k\to\infty} (x^t A^{2k} x)^{1/k} = \lambda_0^2$.

6. Show that the series

$$\sum_{n=1}^{\infty} \frac{\exp(inx)}{n}$$

converges uniformly on any compact subinterval of $(0, 2\pi)$.

Note that $s_N = \sum_{n=0}^{N} e^{inx} = \frac{1-e^{i(N+1)x}}{1-e^{ix}}$, and so for x any compact interval $I \subset (0, 2\pi)$ there is a B such that $|s_N| \leq B$ for all N and all $x \in I$. Now, as in problem 4, summation by parts gives

$$\sum_{n=M+1}^{N} \frac{e^{inx}}{n} = \sum_{n=M+1}^{N-1} s_n\left(\frac{1}{n} - \frac{1}{n+1}\right) + \frac{s_N}{N} - \frac{s_M}{M}.$$

Thus for $x \in I$ and $N > M$

$$\left|\sum_{n=1}^{N} \frac{e^{inx}}{n} - \sum_{n=1}^{M} \frac{e^{inx}}{n}\right| \leq B \sum_{n=M+1}^{N-1} \left(\frac{1}{n} - \frac{1}{n+1}\right) + B\left(\frac{1}{N} + \frac{1}{M}\right) = B\left(\frac{1}{M+1} + \frac{1}{M}\right).$$

From this one sees easily that $S_N(x) = \sum_{n=1}^{N} e^{inx}/n$ is a sequence of functions that converges uniformly on I.