Assignment 3 – due Monday, August 31

1. (Basic S'09) Set $a_0 = 0$ and define a sequence $\{a_n\}_{n=1}^{\infty}$ via the recurrence

$$a_{n+1} = \sqrt{6 + a_n} \qquad \text{for all } n \geq 1.$$

Show that this sequence converges and determine its limit.

The next three are adapted from Rudin, pp.78-79.

2. Determine whether $\sum a_n$ converges for the following choices of a_n.
a) $a_n = (n^{1/n} - 1)^n$
b) $a_n = (1 + z^n)^{-1}$. The answer depends on the complex number z.

3. Suppose that $a_n \geq 0$ and $\sum a_n = \infty$, i.e. the partial sums are unbounded. Show that

$$\sum \frac{a_n}{1 + a_n} = \infty.$$

What about

$$\sum \frac{a_n}{1 + n^2 a_n} \text{ and } \sum \frac{a_n}{1 + n a_n}?$$

4. Suppose that $\sum a_n$ converges, $\{b_n\}$ is monotonic increasing and bounded above. Show that $\sum a_n b_n$ converges. [This is not as easy as it might sound remember that these are not necessarily series with nonnegative terms.]

5. (Basic S'07) Suppose that A is a symmetric, real, $n \times n$ matrix with eigenvalues $\lambda_1 \leq \lambda_2 \leq \cdots \leq \lambda_n$. Find the sets

$$X = \{x \in \mathbb{R}^n : \lim_{k \to \infty} (x^t A^{2k} x)^{1/k} \text{ exists } \}, \text{ and}$$

$$L = \{ \lim_{k \to \infty} (x^t A^{2k} x)^{1/k} : x \in X \}.$$

You get to assume everything that you know about linear algebra here.

6. Show that the series

$$\sum_{n=1}^{\infty} \frac{\exp(inx)}{n}$$

converges uniformly on any compact subinterval of $(0, 2\pi)$.