**Assignment 3 – due Friday, August 31**

1. S'07 Basic Exam # 10. This one is a bit challenging: all of the hypotheses i)-iv) are necessary.

2. S'07 Basic Exam # 4. This looks like linear algebra at first glance. It is really a mix of a little bit of linear algebra with a lot of analysis.

3. S'07 Basic Exam # 12. The metric here is $d(x,y) = ||x-y||$.

4. F'04 Basic Exam #1 or S'05 Basic Exam #5 in Analysis section. This is a standard result. I hope that you haven't seen it.

5. W'06 Basic Exam #2.

6. F'06 Basic Exam #3.

7. S'04 Basic Exam #6. This is another one that mixes linear algebra and analysis, but is mostly analysis. Remember that "$||x|$" is a norm, and try the one-dimensional case, $n = 1$, first. The people writing the problem seemed to have difficulty writing $||x|$, and various other choices of vertical bars appear in the statement. All norms except $||x||_2$ are supposed to be $||x|$.

8. (Rudin, p.166) Prove that the series

$$\Sigma_{n=1}^{\infty}(-1)^n \frac{x^2+n}{n^2}$$

converges uniformly on every bounded interval, but does not converge absolutely for any value of $x$.

**Root Test, Ratio Test and Decreasing Sequences**

On Thursday I discussed the topics above in a lecture that was not very focused. Here is the essence of what I was trying to say.

**Theorem 1**. The series $\Sigma_{n=1}^{\infty} a_n$ is convergent if $\overline{\lim}\ |a_n|^{1/n} < 1$, divergent if $\overline{\lim}\ |a_n|^{1/n} > 1$, and can be either convergent or divergent if $\overline{\lim}\ |a_n|^{1/n} = 1$.

The proof of Theorem 1 is a direct application of the following: if $\overline{\lim}\ b_n = L < \infty$, then, for any $\epsilon > 0$, $b_n > L + \epsilon$ for only finitely many $n$, but $b_n > L - \epsilon$ for infinitely many $n$.

**Theorem 2**. If $a_n \neq 0$ for $n$ sufficiently large, then one has

$$\underline{\lim}\ |\frac{a_{n+1}}{a_n}| \leq \underline{\lim}\ |a_n|^{1/n} \leq \overline{\lim}\ |a_n|^{1/n} \leq \overline{\lim}\ |\frac{a_{n+1}}{a_n}|.$$

The proof of Theorem 2 uses the property of $\overline{\lim}$ used in proving Theorem 1, the corresponding property of $\underline{\lim}$, and $\lim_{n\to\infty} p^{1/n} = 1$ for $p > 0$.

**Corollary (to Theorem 1).** Let $1/R = \lim\ |a_n|^{1/n}$ with $R = 0$ if $\lim\ |a_n|^{1/n} = \infty$ and $R = \infty$ if $\lim\ |a_n|^{1/n} = 0$. Then the power series

$$\Sigma_{n=0}^{\infty} a_n x^n$$

converges for $|x| < R$ and diverges for $|x| > R$.

**Corollary (to Theorems 1 and 2)**. $\Sigma_{n=1}^{\infty} a_n$ converges if $\overline{\lim}\ |\frac{a_{n+1}}{a_n}| < 1$ and diverges if $\underline{\lim}\ |\frac{a_{n+1}}{a_n}| > 1$.

Note that this corollary implies the "undergraduate ratio test": $\Sigma_{n=1}^{\infty} a_n$ converges if $\lim\ |\frac{a_{n+1}}{a_n}| < 1$ and diverges if $\lim\ |\frac{a_{n+1}}{a_n}| > 1$.

If $a_n$ is a decreasing nonnegative sequence, i.e. $a_n \geq a_{n+1} \geq 0$, then one can check that if $n < 2^{m+1}$

$$\Sigma_{k=1}^{n} a_k \leq \Sigma_{k=1}^{m} 2^k a_{2^k}, \text{ and if } n \geq 2^m,\ \Sigma_{k=1}^{n} a_k \geq a_1 + \Sigma_{k=1}^{m} 2^{k-1} a_{2^k}.$$

From this one immediately deduces the handy convergence criterion (discovered by Cauchy):

**Theorem 3** (Rudin, Theorem 3.27, and Tao, Prop.7.3.4). If $a_n \geq a_{n+1} \geq 0$, then $\Sigma a_n$ converges if and only if $\Sigma\ 2^n a_{2^n}$ converges.

For instance, $\Sigma\ n^{-p}$ converges if and only if the geometric series $\Sigma\ 2^{-(p-1)n}$ converges, i.e. if and only if $p > 1$, and $\Sigma(n(\ln n)^p)^{-1}$ converges if and only if the $p$-series $\Sigma\ (n \ln 2)^{-p}$ converges, i.e. if and only if $p > 1$.