Solutions and Comments for Assignment 2

1. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and one-to-one, so that its inverse g is defined on the range of f. Suppose also that $f'(0)$ is not invertible. Prove that g cannot be differentiable at $f(0)$.

The idea here is that f should be a function like $f(x) = x^3$ with $f'(0) = 0$. Sure enough, its inverse $g(y) = y^{1/3}$ is not differentiable at $y = 0$. This problem asks you to prove that this is true in general. To prove that by contradiction suppose that the inverse function *is* differentiable at $y = f(0)$. Our definition of differentiable assumes that g is defined on $B(f(0), r)$ for some $r > 0$. So I will assume this without trying to prove that the range of f contains $B(f(0), r)$: if it doesn't, we have already proven that g is not differentiable at $f(0)$. Then we have $g(f(x)) = x$ for $x \in B(0, s)$ for some $s > 0$, and the chain rule implies $g'(f(0))f'(0) = I$ which says that $f'(0)$, which was assumed to be not invertible, has an inverse. This contradiction is all you need.

2. Let $f : \mathbb{R}^n \to R$ be differentiable. If for all $t > 0$ one has $f(tx) = tf(x)$ for all $x \in \mathbb{R}^n$, show that $f(x) = \nabla f(0) \cdot x$. However, if you take any function g which is differentiable at all points on the sphere $\{|x| = 1\}$, and define $f(0) = 0$ and $f(x) = |x|g(x/|x|)$ for $x \neq 0$, f will have (one-sided) directional derivatives in all directions at $x = 0$.

This is quite easy. Just let x be any fixed point in $\mathbb{R}^n$, differentiate both sides of $f(tx) = tf(x)$ with respect to t, and take the limit as $t \to 0$. The left hand side requires using the chain rule. The example was supposed to illustrate the way that this result fails if you do not assume differentiability of f: in the example f is differentiable at all points except $x = 0$, and even at $x = 0$ it has directional derivatives in all directions, $D_v f(0) = g(v)$.

3. (This one is from Edwards, p.89) Let $w = f(x, y, z)$ and $z = g(x, y)$. Then by the chain rule

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}. \qquad (*)$$

Simplifying this, since $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, you can conclude $\frac{\partial w}{\partial z}\frac{\partial z}{\partial x} = 0$. However, for $w = x + y + z$ and $z = g(x, y) = x + y$ we have $\frac{\partial w}{\partial z} = \frac{\partial z}{\partial x} = 1$ and $(*)$ gives 0=1. Where *exactly* is the mistake in this?

This is just a simple mistake: the functions $w(x, y, z)$ and $w(x, y, g(x, y))$ are both denoted by w. This would be OK if one did not go on to identify the x-derivative of the first:

$$\frac{\partial w}{\partial x}(x, y, z)$$

with the x-derivative of the second:

$$\frac{\partial w}{\partial x}(x, y, g(x, y)) + \frac{\partial w}{\partial z}(x, y, g(x, y))\frac{\partial g}{\partial x}(x, y).$$

4. (Basic S'07) Suppose that the real-valued functions $\{f_n\}$ are twice continuously differentiable on $[0,1]$ and satisfy

$$\lim_{n\to\infty} f_n(x) = f(x),\ |f_n'(x)| \leq 1,\ \text{and } |f_n''(x)| \leq 1$$

for all $x \in [0,1]$ and $n \geq 1$. Prove that f is continuously differentiable on $[0,1]$.

The derivatives $\{f_n'\}$ are an equicontinuous family, and the Arzela-Ascoli theorem gives you a subsequence $\{f_{n_k}'\}_{k=1}^{\infty}$ converging uniformly to $g \in C([0,1])$. For the rest use

$$f_{n_k}(x) - f_{n_k}(0) = \int_0^x f_{n_k}'(t)dt.$$

When you take the limit as $k \to \infty$, the uniform convergence of the f_{n_k}' implies the convergence of the integrals. Hence

$$u(x) - u(0) = \int_0^x g(t)dt,$$

and u is continuously differentiable.

5. Consider the following function on $\mathbb{R}^2$:

$$f(x,y) = \begin{cases} x^2 + y^2, & \text{if x and y are rational numbers} \\ 0 & \text{elsewhere} \end{cases}$$

At what points is f continuous? At what points is f differentiable?

Since the points with rational coordinates are dense and the points with irrational coordinates are also dense, f is only continuous at (x,y)=(0,0). However, directly from the definition, you can see that it is differentiable there too with $\nabla f(0,0) = (0,0)$.

6. (Basic S '06) Let W be the set of continuous real-valued functions on [0,1] satisfying

$$|f(x) - f(y)| \leq |x-y| \text{ and } \int_0^1 (f(x))^2 dx \leq 1.$$

a) Show that there is an M such that $|f(x)| \leq M$ for all $x \in [0,1]$ and all $f \in W$.
b) Show that W is a compact subset of the set of continuous real-valued functions on [0,1], considered as a metric space with $d(f,g) = \max_{x\in[0,1]} |f(x) - g(x)|$.

This problem is the same as problem 5 on the first assignment! In a desperate attempt to justify this glitch I point out that this prepares for the frequently recycled problems on the Basic Exam. The recycling here consisted of replacing $\int_0^1 f(x)dx = 1$ by $\int_0^1 (f(x))^2 dx \leq 1$. That really makes no difference, since, combined with $|f(x) - f(y)| \leq |x-y|$, it still implies that $|f(x)| \leq 2$. That was the use of $\int_0^1 f(x)dx = 1$ in the solution of problem 5.

7. Suppose that F is continuously differentiable on $\mathbb{R}^3$, $F(x_0, y_0, z_0) = 0$ and each component of $\nabla F(x_0, y_0, z_0)$ is nonzero. The the implicit function theorem gives

three functions f, g and h such that $z = f(x,y)$, $y = h(x,z)$ and $x = g(y,z)$ all define the surface $F(x,y,z) = 0$ near (x_0, y_0, z_0). Prove the relation

$$\frac{\partial g}{\partial y}(y,z)\frac{\partial h}{\partial z}(x,z)\frac{\partial f}{\partial x}(x,y) = -1.$$

This is a computation with the chain rule. Differentiate $F(x,y,f(x,y)) \equiv 0$ with respect to x, $F(x,h(x,z),z) \equiv 0$ with respect to z, and $F(g(y,z),y,z) \equiv 0$ with respect to y:

$$\frac{\partial F}{\partial x}(x,y,f(x,y)) + \frac{\partial F}{\partial z}(x,y,f(x,y))\frac{\partial f}{\partial x}(x,y) = 0$$

$$\frac{\partial F}{\partial z}(x,h(x,z),z) + \frac{\partial F}{\partial y}(x,h(x,z),z)\frac{\partial h}{\partial z}(x,z) = 0$$

$$\frac{\partial F}{\partial y}(g(y,z),y,z) + \frac{\partial F}{\partial x}(g(y,z),y,z)\frac{\partial g}{\partial y}(y,z) = 0$$

When you sort out the results, you get the identity. One important point: by the uniqueness in the implicit function theorem $(x,y,f(x,y))$, $(x,h(x,y)x,z)$ and $(g(y,z),y,z)$ are all the same point when (x,y,z) is in a sufficiently small ball centered at (x_0, y_0, z_0).

8. (Basic S'02) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable with $\nabla f(0,0,0) \neq 0$. Show that there exist continuously differentiable g and h defined near (0,0,0) such that $F(x,y,z) = (f(x,y,z), g(x,y,z), h(x,y,z))$ is one-to-one on some neighborhood of (0,0,0).

This problem may seem odd, but it is very easy: if $\frac{\partial f}{\partial x}(0,0,0) \neq 0$, you can choose vectors v and w so that $\{\nabla f(0,0,0), v, w, \}$ is a basis for $\mathbb{R}^3$. Then take $g = v\cdot(x,y,z)$ and $h = w\cdot(x,y,z)$. Defining $F(x,y,z) = (f(x,y,z), g(x,y,z), h(x,y,z))$, the Jacobian matrix $F'(0,0,0)$ will be the matrix with rows $\nabla f(0,0,0)$, v and w. Thus the inverse function theorem applies and shows that F is one-to-one on a ball centered at $(0,0,0)$.