Assignment 2 – due Monday, August 24

1. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and one-to-one, so that its inverse g is defined on the range of f. Suppose also that $f'(0)$ is not invertible. Prove that g cannot be differentiable at $f(0)$.

2. Let $f : \mathbb{R}^n \to R$ be differentiable. If for all $t > 0$ one has $f(tx) = tf(x)$ for all $x \in \mathbb{R}^n$, show that $f(x) = \nabla f(0) \cdot x$. However, if you take any function g which is differentiable at all points on the sphere $\{|x| = 1\}$, and define $f(0) = 0$ and $f(x) = |x|g(x/|x|)$ for $x \neq 0$, f will have (one-sided) directional derivatives in all directions at $x = 0$.

3. (This one is from Edwards, p.89) Let $w = f(x, y, z)$ and $z = g(x, y)$. Then by the chain rule

$$\frac{\partial w}{\partial x} = \frac{\partial w}{\partial x}\frac{\partial x}{\partial x} + \frac{\partial w}{\partial y}\frac{\partial y}{\partial x} + \frac{\partial w}{\partial z}\frac{\partial z}{\partial x}. \qquad (*)$$

Simplifying this, since $\frac{\partial x}{\partial x} = 1$ and $\frac{\partial y}{\partial x} = 0$, you can conclude $\frac{\partial w}{\partial z}\frac{\partial z}{\partial x} = 0$. However, for $w = x + y + z$ and $z = g(x, y) = x + y$ we have $\frac{\partial w}{\partial z} = \frac{\partial z}{\partial x} = 1$ and $(*)$ gives 0=1. Where *exactly* is the mistake in this?

4. (Basic S'07) Suppose that the real-valued functions $\{f_n\}$ are twice continuously differentiable on $[0, 1]$ and satisfy

$$\lim_{n\to\infty} f_n(x) = f(x),\ |f_n'(x)| \leq 1, \text{ and } |f_n''(x)| \leq 1$$

for all $x \in [0, 1]$ and $n \geq 1$. Prove that f is continuously differentiable on $[0, 1]$.

5. Consider the following function on $\mathbb{R}^2$:

$$f(x, y) = \begin{cases} x^2 + y^2, & \text{if x and y are rational numbers} \\ 0 & \text{elsewhere} \end{cases}$$

At what points is f continuous? At what points is f differentiable?

6. (Basic S '06) Let W be the set of continuous real-valued functions on [0,1] satisfying

$$|f(x) - f(y)| \leq |x - y| \text{ and } \int_0^1 (f(x))^2 dx \leq 1.$$

a) Show that there is an M such that $|f(x)| \leq M$ for all $x \in [0, 1]$ and all $f \in W$.
b) Show that W is a compact subset of the set of continuous real-valued functions on [0,1], considered as a metric space with $d(f, g) = \max_{x\in[0,1]} |f(x) - g(x)|$.

7. Suppose that F is continuously differentiable on $\mathbb{R}^3$, $F(x_0, y_0, z_0) = 0$ and each component of $\nabla F(x_0, y_0, z_0)$ is nonzero. The the implicit function theorem gives three functions f, g and h such that $z = f(x, y)$, $y = h(x, z)$ and $x = g(y, z)$ all define the surface $F(x, y, z) = 0$ near (x_0, y_0, z_0). Prove the relation

$$\frac{\partial g}{\partial y}(y, z)\frac{\partial h}{\partial z}(x, z)\frac{\partial f}{\partial x}(x, y) = -1.$$

8. (Basic S'02) Suppose $f : \mathbb{R}^3 \to \mathbb{R}$ is continuously differentiable with $\nabla f(0, 0, 0) \neq 0$. Show that there exist continuously differentiable g and h defined near (0,0,0) such that $F(x, y, z) = (f(x, y, z), g(x, y, z), h(x, y, z))$ is one-to-one on some neighborhood of (0,0,0).