Matrices with Distinct Eigenvalues

Suppose that A is an $n \times n$ matrix with eigenvalues $r_1, r_2, \ldots, r_n$ which are all different, i.e. $r_i = r_j$ only if $j = i$. [This is the same as assuming that the characteristic polynomial det(A-rI) has distinct roots.] Then we can choose nonzero eigenvectors $\underline{v}_1$, $\underline{v}_2, \ldots, \underline{v}_n$ for each eigenvalue. The topic of this note is the following basic fact: we can write ANY n-component vector as a linear combination of $\underline{v}_1, \underline{v}_2, .., \underline{v}_{n-1}$ and $\underline{v}_n$. This is the basis for our claim that the general solution to $\underline{x}' = A\underline{x}$ is

$$\underline{x} = c_1 e^{r_1 t}\underline{v}_1 + c_2 e^{r_2 t}\underline{v}_2 + \cdots + c_n e^{r_n t}\underline{v}_n.$$

It amounts to saying that we can solve the system of equations

$$c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_n\underline{v}_n = \begin{pmatrix} a_1 \\ a_2 \\ | \\ a_n \end{pmatrix}$$

for every choice of $a_1, a_2, \ldots, a_n$. We can write these equations in matrix form using the matrix whose columns are the $\underline{v}_j$'s:

$$\begin{pmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ | \\ c_n \end{pmatrix} = \begin{pmatrix} a_1 \\ a_2 \\ | \\ a_n \end{pmatrix}. \tag{1}$$

Now we can use a standard linear algebra chain of reasoning. The system of equations (1) can be solved if the matrix

$$\begin{pmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{pmatrix}$$

has an inverse. It has an inverse if its determinant is not zero. Its determinant is not zero if the ONLY solution to

$$\begin{pmatrix} | & | & & | \\ \underline{v}_1 & \underline{v}_2 & \cdots & \underline{v}_n \\ | & | & & | \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ | \\ c_n \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ | \\ 0 \end{pmatrix}$$

is $c_1 = c_2 = \cdots = c_n = 0$. So we need to show that when

$$c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_n\underline{v}_n = \underline{0}, \tag{2}$$

all the c_j's must be zero.

The proof that (2) implies all the c_j's are zero really shows why eigenvectors are so useful. Pick a j_0. We are going to show that $c_{j_0} = 0$. Multiply $c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_n\underline{v}_n$ by $A - r_kI$ where $k \neq j_0$. That gives

$$(A - r_kI)(c_1\underline{v}_1 + c_2\underline{v}_2 + \cdots + c_n\underline{v}_n) = c_1(A - r_kI)\underline{v}_1 + c_2(A - r_kI)v_2 + \cdots + c_n(A - r_kI)v_n$$

$$= (r_1 - r_k)c_1\underline{v}_1 + (r_2 - r_k)c_2\underline{v}_2 + \cdots + (r_n - r_k)c_n\underline{v}_n, \tag{3}$$

but the right hand side of (3) is zero because it is also $(A - r_kI)\underline{0}$. Also note that the coefficient of $\underline{v}_k$ on the right hand side of (3) is zero, since it has the factor $(r_k - r_k)$ in it. So we have eliminated

the term muliplied by c_k. We can now multiply the right hand side of (3) by $(A - r_lI)$ where $l \neq j_0$, eliminating the term multiplied by c_l – and keep on doing this until we have eliminated the terms multiplied by each c_j except c_{j_0}. At that point all we have left is

$$(r_{j_0} - r_1)(r_{j_0} - r_2)\cdot\cdot(r_{j_0} - r_{j_0-1})(r_{j_0} - r_{j_0+1})\cdot\cdot(r_{j_0} - r_n)c_{j_0}\underline{v}_{j_0} = \underline{0}.$$

In this product none of the factors $(r_{j_0} - r_j)$ is zero because j never equals j_0, and $\underline{v}_{j_0} \neq \underline{0}$ because we chose it that way. So we must have $c_{j_0} = 0$. Since we can repeat this argument for every choice of the index j_0, we have completed the proof.