The Method of Undetermined Coefficients – Concise Version

You can always use the "Method of Undetermined Coefficients" to solve

$$y'' + py' + qy = P(t)e^{rt} \tag{1}$$

when p and q are constants and P is a polynomial : $P(t) = a_n t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$. To use the method you assume that $y = Q(t)e^{rt}$, where Q is also a polynomial – with undetermined coefficients. You solve (1) by finding the coefficients for Q that make y a solution. To see how this works substitute $y = Q(t)e^{rt}$ into equation (1). That gives

$$(Q''e^{rt} + 2Q're^{rt} + Qr^2e^{rt}) + p(Q'e^{rt} + Qre^{rt}) + qQe^{rt} = Pe^{rt}.$$

Simplify that a little bit (cancel the e^{rt} factors and group the terms involving Q, Q' and Q'' together), and you have

$$(r^2 + pr + q)Q(t) + (2r + p)Q'(t) + Q''(t) = P(t). \tag{2}$$

. Now there are three cases:

Case I: $r^2 + pr + q \neq 0$. Note that this is case when $y = e^{rt}$ is **not** a solution to the homogeneous equation $y'' + py' + qy = 0$. In this case you take Q to have exactly the same form as P,

$$Q(t) = A_n t^n + A_{n-1}t^{n-1} + \cdots + A_1 t + A_0$$

To solve (2) equate coefficients of powers of t on both sides of (2), starting with the highest powers: equating the coefficients of t^n will give you A_n. Then, with A_n known, equating the coefficients of t^{n-1} will give you A_{n-1}, and you can continue on down, finding A_0 last.

Case II: $r^2+pr+q = 0$ but $2r+p \neq 0$. In this case $y = e^{rt}$ **is** a solution of the homogeneous equation $y'' + py' + qy = 0$, but the characteristic equation has distinct roots. Note that now the highest power in the polynomial Q does not appear in equation (2) since (2) only contains Q' and Q''. So now you have to choose Q to have degree one higher than P which means taking

$$Q(t) = A_{n+1}t^{n+1} + A_n t^n + \cdots + A_1 t + A_0.$$

With that choice you proceed as in case I, equating coefficients to find A_{n+1} first, then A_n and so on down. You do not need to include the constant term A_0 in Q in this case: any choice of A_0 will work because $A_0 e^{rt}$ is a solution of the homogeneous equation.

Case III: $r^2 + pr + q = 0$ and $2r + p = 0$. In this case $y = e^{rt}$ **is** a solution of the homogeneous equation $y'' + py' + qy = 0$, and the general solution to the homogeneous equation is $y = (c_1 + c_2 t)e^{rt}$. Now the highest two powers of Q do not appear in (2) since the lefthand side of (2) is just Q'', and you have to choose Q to have degree two higher than P:

$$Q(t) = A_{n+2}t^{n+2} + A_{n+1}t^{n+1} + A_n t^n + \cdots A_2 t^2 + A_1 t + A_0.$$

and you find the coefficients of Q by equating coefficients on both sides of (2) as before. This time you do not need to include $A_1 t + A_0$ – any choice of A_1 and A_0 will work because $(A_1 t + A_0)e^{rt}$ is a solution of the homogeneous equation.

That's all there is to it, and you can use complex numbers for r. That means that you can solve

$$y'' + py' + qy = P(t)e^{at}\cos(bt) \text{ and } y'' + py' + qy = P(t)e^{at}\sin(bt)$$

by taking real and imaginary parts of your solution – as long as p and q are real numbers and $P(t)$ has real coefficients. How everything works will be a lot clearer after you **practice**!