NAME

HOUR EXAM II – MATHEMATICS 33B

November 19, 2004

1.(2 pts) a) Define what it means for a number S to be the sum of the infinite series $\sum_{n=1}^{\infty} a_n$.

Solution. Had I known that anyone would get this wrong I would have made it worth more points. Infinite sums are defined as limits of their partial sums. If the partial sums do not have a limit, the infinite sum is undefined. End of story. Please learn this. Symbolically it goes this way. The partial sum S_n is defined by $S_n = \sum_{k=1}^{n} a_k$. The infinite sum $\sum_{n=1}^{\infty} a_n$ is defined to be $\lim_{n\to\infty} S_n$ when that limit exists. Otherwise it is undefined.

b) (8 pts) Prove that if $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$.

Solution. Since $a_n = S_n - S_{n-1}$ by the definition of partial sums in part a), and the hypothesis here is that $\lim_{n\to\infty} S_n = S$ exists, we have

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

That argument used the limit theorem that $\lim_{n\to\infty}(A_n - B_n) = \lim_{n\to\infty} A_n - \lim_{n\to\infty} B_n$, when both limits on the right hand side of equation exist. This is all that was required on the hour exam.

Still, one could ask why $\lim_{n\to\infty} S_n = S$ implies $\lim_{n\to\infty} S_{n-1} = S$. The answer is that $\lim_{n\to\infty} S_n = S$ means that, for any number $\epsilon > 0$, if you go far enough out in the sequence, i.e. to $n > N_\epsilon$ you will have $|S_n - S| < \epsilon$. That implies that for $n > N_\epsilon + 1$ you will have $|S_{n-1} - S| < \epsilon$ for any $\epsilon > 0$, and proves that $\lim_{n\to\infty} S_{n-1} = S$. This is sort of proof where you have to follow the logic very closely to see that there is anything to prove! Which is why I did not require it on the exam.

2. a) (8 pts) Find the Taylor series for $(1-x)^{-2/3}$ about $x=0$, that is find the coefficients a_n such that

$$(1-x)^{-2/3} = 1 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots + ...$$

when x is in the interval of convergence. Find the general term in this series – you are allowed to use $\cdots$ in the middle of long products of integers.

Solution. For $f(x) = (1-x)^{-2/3}$ one has $f^{(1)}(x) = (2/3)(1-x)^{-5/3}$, $f^{(2)}(x) = (5/3)(2/3)(1-x)^{-8/3}, \ldots, f^{(n)}(x) = ((3n-1)/3)\cdots(5/3)(2/3)(1-x)^{-(3n+2)/3}$. So we have

$$a_n = \frac{f^{(n)}(0)}{n!} = \frac{(3n-1)(3(n-1)-1)\cdots(8)(5)(2)}{3^n n!}.$$

That was all that was asked for, but the radius of convergence is easy to find, too.

$$\lim_{\to\infty} \frac{|a_{n+1}x^{n+1}|}{|a_n x^n|} = \lim_{n\to\infty} \frac{(3n+2)}{3(n+1)}|x| = |x|.$$

So the radius of convergence is 1 – as you might expect since $f(1)$ is undefined.

b) (8 pts) Find the radius of convergence of the power series $\sum_{n=0}^{\infty}(3^n + 2^n)x^{2n}$

Solution. By the ratio test

$$\lim_{n\to\infty} \frac{(3^{n+1} + 2^{n+1})|x|^{2n+2}}{(3^n + 2^n)|x|^{2n}} = \lim_{n\to\infty} \frac{3 + 2(2/3)^n}{1 + (2/3)^n}|x|^2 = 3|x|^2$$

So the series converges for $3|x|^2 < 1$ and diverges for $3|x|^2 > 1$. In other words the radius of convergence is $1/\sqrt{3}$. Note that in computing the limit I followed my own advice from Wednesday's class and DIVIDED NUMERATOR AND DENOMINATOR BY THE LARGEST TERM IN THE DENOMINATOR which happened to be $3^n|x|^{2n}$ in this case.

3. (10 pts) Find the general solution to $y'' + y = te^{2t} + \cos 2t$.

Solution. There are several ways to do this. You can find particular solutions for the inhomogeneous terms te^{2t} and $\cos 2t$ separately and then add them together, or you can find them in one step if you use enough undetermined coefficients. I will do it the second way.

Since the roots of the characteristic equation ($r^2+1 = 0$) for the homogeneous equation are $r = \pm i$, neither e^{2t} nor $\cos 2t$ is a solution of the homogeneous equation, and we can try

$$y = (At + B)e^{2t} + C \cos 2t + D \sin 2t$$

as a particular solution. Actually, since there is is no multiple of y' in $y'' + y$, we will not need D – it will just turn out to be 0. Nonetheless we require

$$((At + B)e^{2t} + C \cos 2t + D \sin 2t)'' + ((At + B)e^{2t} + C \cos 2t + D \sin 2t) = te^{2t} + \cos 2t.$$

After a little differentiation that becomes

$$4(At+B)e^{2t} + 4Ae^{2t} - 4C \cos 2t - 4D \sin 2t + (At+B)e^{2t} + C \cos 2t + D \sin 2t = te^{2t} + \cos 2t.$$

Equating the coefficients of the four functions, $\cos 2t$,$\sin 2t$, te^{2t} and e^{2t} on both sides of this equation gives

$$-3C = 1, \ -3D = 0, \ 5A = 1, \ \text{and } 5B + 4A = 0.$$

That system you can solve by looking at it. Since the general solution to the homogeneous equation is $y = c_1 \cos t + c_2 \sin t$, the general solution here is

$$y = (\frac{t}{5} - \frac{4}{25})e^{2t} - \frac{1}{3} \cos 2t + c_1 \cos t + c_2 \sin t.$$

4.(10 pts) Find a power series

$$y = \sum_{n=0}^{\infty} a_n x^n$$

for the solution of $y'' + x^2y' + 2xy = 0$ satisfying $y(0) = 1$ and $y'(0) = 0$. You do not need to find the general term in the power series, but you should find the general recursion relation for the coefficients a_n and the coefficients a_2, a_3 and a_{17}.

Solution Directly from the formula for the differentiation of power series we have

$$y'' + x^2y' + 2xy = \sum_{n=2}^{\infty} n(n-1)a_n x^{n-2} + \sum_{n=1}^{\infty} na_n x^{n+1} + \sum_{n=0}^{\infty} 2a_n x^{n+1} \tag{1}$$

Notice that I have moved the factors of x^2 on y' and $2x$ on y inside the summations. Not doing that leads to disaster.

The next step is crucial: you have to get the power of x the same in each sum in formula (1) before you equate coefficients. I think the best way to do this is to substitute $m = n - 2$ in the first sum, and $m = n + 1$ in the second and third. That gives

$$y'' + x^2y' + 2xy = \sum_{m=0}^{\infty} (m+2)(m+1)a_{m+2}x^m + \sum_{m=2}^{\infty} (m-1)a_{m-1}x^m + \sum_{m=1}^{\infty} 2a_{m-1}x^m \tag{2}$$

Now for each m you set the coefficient of x^m equal to zero – because all the coefficients in the power series for $0 = y'' + x^2y' + 2xy$ are zero.

(m=0) The only place you have an x^0 term is in the first sum, and its coefficient is $(2)(1)a_2$ – so $a_2 = 0$.

(m=1) The coefficients of x^1 in the first and last sums in formula (2) add to $(3)(2)a_3 + 2a_0$. So that's 0 and you have $a_3 = -a_0/3$, but $a_0 = y(0) = 1$, so $a_3 = -1/3$.

($m \geq 2$) For $m \geq 2$ all three sums have x^m in them, and equating the coefficient of x^m to zero gives the recursion relation

$$(m+2)(m+1)a_{m+2} + (m-1)a_{m-1} + 2a_{m-1} = 0 \text{ or } a_{m+2} = \frac{-a_{m-1}}{m+2}$$

Note that this says that a_{m-1} determines a_{m+2}. So $a_2 = 0 \to a_5 = 0 \to a_8 = 0 \to \cdots \to a_{17} = 0$.

This is all that was asked for in the problem, but you can go further. Since $a_1 = y'(0) = 0$ and $a_2 = 0$, only the coefficients a_{3n}, $n = 0, 1, 2, \ldots$, are going to be nonzero, and the general recursion relation says

$$a_{3n} = \frac{-a_{3(n-1)}}{3n} \text{ so } a_6 = \frac{-a_3}{6} = \frac{1}{6 \cdot 3},\ a_9 = \frac{-1}{9 \cdot 6 \cdot 3},$$

and so on, until you see that

$$a_{3n} = \frac{(-1)^n}{(3n)(3n-3)\cdots(3)} = \frac{(-1)^n}{3^n n!}.$$

Comparing that to the power series for e^x, you can figure out that $y = e^{-x^3/3}$. This really is the solution to the differential equation with $y(0) = 1$ and $y'(0) = 0$. Try it!

5. (10 pts) In this problem you will use the method of variation of parameters to find a formula for the solution to $y'' = -g$, satisfying $y(0) = y(1) = 0$, for a general continuous function g.

a) Choose a fundamental set of solutions to $y'' = 0$ with $y_1(0) = 0$ and $y_2(1) = 0$.

Solution. I knew this problem was a little difficult – but I didn't think this *part* of it would be difficult. Don't you recall that the general solution to $y'' = 0$ is $y = ax + b$? I have to admit that our general methods do not give this answer easily. If you use the characteristic equation method, you get $r^2 = 0$ which means $r = 0$ is a double root, so the general solution is $y = c_1 e^0 + c_2 x e^0$ which is a fancy way of writing $y = c_1 + c_2 x$. Anyway a fundamental set that works here is $y_1 = x$ and $y_2 = 1 - x$.

b) Use the fundamental set from a) to solve $y'' = -g$ by the method of variation of parameters, that is, look for y in the form $y(x) = c_1(x)y_1(x) + c_2(x)y_2(x)$. In most problems that we have done c_1 and c_2 have been written as indefinite integrals. Do not do that here. Instead show that you can write c_1 as a definite integral from x to 1 and c_2 as a definite integral from 0 to x.

Solution. Here you just begin following the variation of parameters method with the fundamental set from part a). That means solving

$$\begin{aligned} c_1' y_1 + c_2' y_2 &= 0 \\ c_1' y_1' + c_2' y_2' &= -g \end{aligned}$$

which is

$$\begin{aligned} x c_1' + (1 - x) c_2' &= 0 \\ c_1' - c_2' &= -g \end{aligned}$$

Subtracting the second equation multiplied by x from the first gets you $c_2' = xg$, and substituting this into the second equation gives $c_1' = (x - 1)g$.

Now you need to follow the instructions carefully. Setting

$$c_2(x) = \int_0^x sg(s)ds$$

works because the Fundamental Theorem of Calculus says this makes $c_2'(x) = xg$. For c_1 you need

$$c_1(x) = -\int_x^1 (s - 1)g(s)ds = \int_x^1 (1 - s)g(s)ds$$

because differentiating an integral with respect to its lower limit brings in a minus sign. So the solution is

$$y(x) = x \int_x^1 (1 - s)g(s)ds + (1 - x) \int_0^x sg(s)ds$$

c) Explain why the particular solution that you found in part b) solves the problem stated at the beginning.

There is not much to say here: the construction makes both terms in $y(x)$ equal zero at both $x = 0$ and $x = 1$ and the method of variation of parameters guarantees that $y'' = -g$. I just wanted you to notice that $c_1(1) = c_2(0) = 0$.