Mathematics 33B

This is intended to be an outline for the entire course and a guide for studying for the final exam.

I. First Order Equations

Everything we did here was devoted to the following three topics:

i) Solving $y' + a(x)y = g(x)$ for the unknown function $y(x)$ (Section 2.1).

ii) Solving $y' = f(x)h(y)$ for the unknown function $y(x)$ (Section 2.2).

iii) Learning to use these differential equations in simple applications (Section 2.3).

In all of these we often needed to find the (unique) solution satisfying an "initial value" condition of the form $y(x_0) = y_0$. There is not too much to say here except **practice** doing these things. You should definitely know how to explain that

$$y' + a(x)y = g(x) \text{ is equivalent to } (e^{A(t)}y)' = e^{A(t)}g(t),$$

where $A(t) = \int a(t)dt$ is any anti-derivative of $a(t)$. That is, after all, how you solve i) using an integrating factor. For ii) be on the lookout for values of y where $h(y) = 0$. Remember, if $h(y_0) = 0$, then the constant function $y(x) = y_0$ will be a solution to $y' = f(x)h(y)$ that you will NOT find by the standard method of reducing the equation to $\int (h(y))^{-1}dy = \int f(x)dx$.

For all first order equations remember that two solutions of the same equation **cannot cross** each other: if they did and you labelled the crossing point (x_0, y_0), you would have **two** solutions to the equation satisfying the initial condition $y(x_0) = y_0$. That violates the existence and uniqueness theorem for first order equations. Quite embarassingly, that theorem is not in the syllabus – it is Theorem 2.4.2 from Section 2.4 of Boyce and DiPrima which we did not cover. Its proof is well beyond the scope of this course, but it was stated in lecture, and you should **know what it says**.

II. Second Order Equations

This was a big topic for us. I will break it down into the subtopics

i) Constant Coefficient Homogeneous Equations $y'' + py' + qy = 0$ (Sections 3.1, 3.4 and the "Repeated Roots" part of 3.5)

ii) General Linear Homogeneous Second Order Equations $y'' + p(t)y' + q(t)y = 0$ (Sections 3.2 and 3.3)

iii) Constant Coefficient Inhomogeneous Equations of the form $y'' + py' + qy = f(t)$, where f is a sum of terms of the form $P(t)e^{rt}$ with $P(t)$ a polynomial (Section 3.6)

iv) General Linear Inhomogeneous Second Order Equations $y'' + p(t)y' + q(t)y = g(t)$ (Section 3.7)

v) Applications of Second Order Equations to Vibration Problems (Sections 3.8 and 3.9)

vi) Series Solutions to Homogeneous Second Order Linear Equations (Sections 5.2 and 5.3)

Topic i) is very straightforward: the roots r_1 and r_2 of the characteristic polynomial $r^2 + pr + q$ tell you everything. Remember, since we always assume that p and q are real numbers, there are three cases: real distinct roots, roots $r_1 = a + ib$ and $r_2 = a - ib$, and the double real root case $r_1 = r_2$. The only time I fooled people on this (unintentionally) was on Problem 5 on Hour Exam II where you had to solve $y'' = 0$ – and many people forgot that all this theory just comes down to $y = Ax + B$ for that equation.

Topic ii) is not at all straightforward. Let me try to organize it a little bit. There are really just three main points here:

a) The existence and uniqueness theorem for second order linear equations (Theorem 3.2.1). Know what that says.

b) The following simple, but somehow confusing, fact: if $y_1'' + p(t)y_1' + q(t)y_1 = g_1(t)$ and $y_2'' + p(t)y_2' + q(t)y_2 = g_2(t)$, then for any constants c_1 and c_2, the function $y = c_1y_1 + c_2y_2$ will satisfy the equation $y'' + p(t)y' + q(t)y = c_1g_1(t) + c_2g_2(t)$. This has turned up in lots of places, sometimes with $g_1(t) = g_2(t) = g(t)$ and many times with $g_1(t) = g_2(t) = 0$. It is the essence of linearity. It includes Theorem 3.2.2 (The Principle of Superposition), and it is the key to the infamous Problem 4 on Hour Exam I. Learn the simple computation which is the proof of this fact.

c) The Wronskian and its relation to fundamental sets of solutions. There has been a lot of confusion about the difference between linearly independent vectors and linearly independent functions. It is not necessary to talk about linearly independent functions here.

The **vectors** $(y_1(t_0), y_1'(t_0))$ and $(y_2(t_0), y_2'(t_0))$ are linearly independent if the Wronskian

$$W(t_0) = \det \begin{pmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{pmatrix} \neq 0,$$

and they are linearly dependent (which for two component vectors means one is a scalar multiple of the other) if $W(t_0) = 0$. If they are linearly independent vectors, then you can solve the pair of equations (for the unknowns c_1 and c_2)

$$c_1 y_1(t_0) + c_2 y_2(t_0) = y_0, \;\; c_1 y_1'(t_0) + c_2 y_2'(t_0) = y_0' \tag{1}$$

for every pair of numbers y_0 and y_0'. If they are linearly dependent then there will always be choices of y_0 and y_0' for which the pair of equations (1) has no solution. These are linear algebra facts that really have nothing to do with differential equations. However, if you assume that $y_1(t)$ and $y_2(t)$ are solutions to the same differential equation $y'' + p(t)y' + q(t)y = 0$, a miraculous thing happens. Then the Wronskian $W(t)$ is either never zero or always zero on any interval, $I = \{a < t < b\}$, where $p(t)$ and $q(t)$ are continuous. That comes from Abel's Theorem, $W(t) = C\exp(-\int p(t)dt)$. So if $W(t_0) \neq 0$ for one point t_0 in I, then $W(t) \neq 0$ at every point in I, and for **every** point t_1 in I you can write the solution of the differential equation satisfying $y(t_1) = y_0$ and $y'(t_1) = y_0'$ as $c_1 y_1 + c_2 y_2$. Thus $\{y_1, y_2\}$ is a fundamental set of solutions on I if the Wronskian of y_1 and y_2 is nonzero at one point in I.

This is what I expect you to retain from Sections 3.2 and 3.3.

Topic iii) is the method of undetermined coefficients. The underlying idea here has not received the attention it deserves because it has been easier just to give you rules. What is really going on is this: you have to begin with a set of linear combinations of functions such that any derivative of a function in the set will be in the set, too. There are not a lot of sets like that. Some of the simplest ones are:

$$\{A + Bt + Ct^2\}, \;\; \{A\cos bt + B\sin bt\}, \;\text{and}\; \{Ae^{rt}\}.$$

More generally, for each choice of a, b and n you have the set

$$\{P(t)e^{at}\cos bt + Q(t)e^{at}\sin bt, P(t) \text{ and } Q(t) \text{ general polynomials of degree } n\}.$$

In most cases all you have to do is pick the smallest set of this form that includes the inhomogeneous term in the equation. Unfortunately, when $r = a + ib$ is a simple root (or a double root) of the characteristic polynomial, you have to pick $P(t)$ and $Q(t)$ to have degrees one (or two) higher than the polynomials in the inhomogeneous term. The best way to get good at this is to practice. Do a lot of the exercises in Section 3.6 in Boyce and DiPrima.

Topic iv) is the method of variation of parameters. This one, too, seems to be a little difficult. The idea is a little odd. You look for a solution to the inhomogeneous equation in the form

$$y(t) = c_1(t)y_1(t) + c_2(t)y_2(t),$$

where y_1 and y_2 are a fundamental set of solutions for the homogeneous equation. Then you need to add the condition

$$c_1'(t)y_1(t) + c_2'(t)y_2(t) = 0$$

The point of that condition is that it simplifies the equations that c_1 and c_2 will have to satisfy if y is a solution. The next step is simply to substitute $y = c_1y_1 + c_2y_2$ in the equation

$$y'' + p(t)y' + q(t)y = g(t) \qquad (2).$$

Using y_1 and y_2 are solutions of the homogeneous equation and the condition

$$c_1'(t)y_1(t) + c_2'(t)y_2(t) = 0,$$

you find that (2) will hold if

$$c_1'(t)y_1'(t) + c_2'(t)y_2'(t) = g(t).$$

So you have to solve the system of equations

$$c_1'(t)y_1(t) + c_2'(t)y_2(t) = 0$$

$$c_1'(t)y_1'(t) + c_2'(t)y_2'(t) = g(t),$$

to find c_1' and c_2'. You will always be able to solve it because y_1 and y_2 are a fundamental set of solutions (do you see why?). The method of variation of parameters is more useful as a way of finding integral formulas for solutions, than as a way of solving equations explicitly. That was what you were asked to do in Problem 5 on Hour Exam 2. There are several exercises in Boyce and Diprima Section 3.7 where you are asked to that. Try the ones you have not already done.

Topic v) will not be on the final exam ... explicitly. Just as on the second hour exam I will not give you questions where you need to convert units or set up differential equations for the motion of vibrating systems. However, the equations themselves might turn up, and you might be asked to solve them using undetermined coefficients.

Finally there is topic vi), solving differential equations with power series. The only thing to remember here is to be as methodical as possible. If you have any difficulties just follow the procedure in the solution of problem 21 in Section 7.2 or problem 4 on Hour Exam II. This material is in Section 5.2 of Boyce and DiPrima. From the additional material in Section 5.3, finding the first few terms in a power series solution by differentiating the differential equation was covered in the course, but the Theorem 5.3.1 and the material following it were not.

III Infinite Series

This was, as those of you who have seen infinite series before know, a very limited introduction to infinite series. It was directed quite narrowly toward power series which made the geometric series the natural starting point. We did a lot with geometric series, including using them to find the power series

$$\ln(1-x) = \sum_{n=1}^{\infty} \frac{x^n}{n} \qquad \text{and} \qquad \arctan x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1}.$$

However, the main theoretical points had to do with general series. They were the definition $\sum_{n=1}^{\infty} a_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k$ – which surprisingly few of you knew on Hour Exam II – and the following.

i) If $\sum_{n=1}^{\infty} a_n$ is convergent, then $\lim_{n\to\infty} a_n = 0$.

Proof. Let $S_n = \sum_{k=1}^{n} a_k$. Then $a_n = S_n - S_{n-1}$. So $\lim a_n = \lim(S_n - S_{n-1})$. $\lim_{n\to\infty} S_n = S$ exists by the hypothesis that $\sum_{n=1}^{\infty} a_n$ is convergent. To see that

$\lim_{n\to\infty} S_{n-1} = \lim_{n\to\infty} S_n = S$ you could argue from the definition of limit, as in the solutions to Hour Exam II, but it is simpler and acceptable to just change variables: let $m = n-1$; then $\lim_{n\to\infty} S_{n-1} = \lim_{(m+1)\to\infty} S_m$, and letting $m+1$ go to infinity is definitely the same as letting m go to infinity. So

$$\lim_{m+1\to\infty} S_m = \lim_{m\to\infty} S_m = S$$

So

$$\lim_{n\to\infty} a_n = \lim_{n\to\infty} (S_n - S_{n-1}) = \lim_{n\to\infty} S_n - \lim_{n\to\infty} S_{n-1} = S - S = 0$$

Note that "the limit of a difference is the difference of the limits" was used to get the second equality in that.

ii) The "Least Upper Bound Axiom": if $\mathcal{S}$ is a set of real numbers and there is a number $B < \infty$ such that $x \leq B$ for every $x \in \mathcal{S}$, then there is a real number M such that $x \leq M$ for every $x \in \mathcal{S}$ AND for any $M' < M$ some number in $\mathcal{S}$ is greater than M'.

Naturally enough M is called the least upper bound of $\mathcal{S}$. This was an axiom for us. You can prove it using more fundamental properties of real numbers, but it takes a while.

iii) If $a_n \geq 0$ for all n, and there is a number $B < \infty$ such that the partial sums $S_n \leq B$ for all n, then $\sum_{n=1}^{\infty} a_n$ is convergent.

Proof. Since $a_n \geq 0$ for all n, we have $S_{n-1} \leq S_n$ for all n, i.e. the partial sums form a nondecreasing sequence. Since the partial sums are bounded by B, they have a least upper bound M. We will show that they converge to M. Given $\epsilon > 0$, one of the partial sums, S_N, must be greater than $M - \epsilon$ or M would not be the LEAST upper bound. So $S_N > M - \epsilon$, and that implies $S_n > M - \epsilon$ for $n \geq N$ because the sequence of partial sums is nondecreasing. However, the partial sums also satisfy $S_n \leq M$ for all n, because M is their least UPPER BOUND. So we have the inequalities

$$M - \epsilon < S_n \leq M$$

for all $n \geq N$. Since ϵ is an arbitrary positive number, we have shown

$$\lim_{n \to \infty} S_n = M$$

by the definition of limit. Therefore $\sum_{n=1}^{\infty} a_n = M$ by the definition i).

iv) If $\sum_{n=1}^{\infty} |a_n|$ is convergent, then $\sum_{n=1}^{\infty} a_n$ is convergent ("absolute convergence implies convergence").

Proof. The main step here is separating the positive and negative terms. You do that this way: let

$$a_n^+ = \begin{cases} a_n \text{ if } a_n \geq 0 \\ 0 \text{ if } a_n < 0 \end{cases} \text{ and } a_n^- = \begin{cases} 0 \text{ if } a_n \geq 0 \\ -a_n \text{ if } a_n < 0 \end{cases}.$$

Then both a_n^+ and a_n^- are nonnegative, and we have $a_n^+ + a_n^- = |a_n|$ and $a_n^+ - a_n^- = a_n$. Let $B = \sum_{n=1}^{\infty} |a_n|$. Remember that B is the least upper bound of the partial sums of

$\sum_{n=1}^{\infty} |a_n|$. The number B is also an upper bound for all the partial sums of $\sum_{n=1}^{\infty} a_n^+$:

$$\sum_{k=1}^{n} a_k^+ \leq \sum_{k=1}^{n} |a_k| \leq B$$

and an upper bound for all the partial sums of $\sum_{n=1}^{\infty} a_n^-$:

$$\sum_{k=1}^{n} a_k^- \leq \sum_{k=1}^{n} |a_k| \leq B$$

Therefore **by iii)** both $\sum_{n=1}^{\infty} a_n^+$ and $\sum_{n=1}^{\infty} a_n^-$ are convergent and equal to the limits of their partial sums which we will call S^+ and S^-. We have

$$\lim_{n\to\infty} \sum_{k=1}^{n} a_k = \lim_{n\to\infty} \sum_{k=1}^{n} (a_k^+ - a_k^-) = \lim_{n\to\infty} (\sum_{k=1}^{n} a_k^+ - \sum_{k=1}^{n} a_k^-)$$

$$= \lim_{n\to\infty} \sum_{k=1}^{n} a_k^+ - \lim_{n\to\infty} \sum_{k=1}^{n} a_k^- = S^+ - S^-.$$

This shows that the limit of the partial sums of $\sum_{n=1}^{\infty} a_n$ exists, so $\sum_{n=1}^{\infty} a_n$ is convergent. Note that we used "the limit of a difference is the difference of the limits" again in that last sequence of equalities (where?).

v). The ratio test: $\sum a_n$ is absolutely convergent if $\lim_{n\to\infty} |a_{n+1}/a_n| = L < 1$ and divergent if $\lim_{n\to\infty} |a_{n+1}/a_n| = L > 1$. Note that for practice in using the ratio test you can use problems 1-8 in Section 5.1 of Boyce and DiPrima to supplement the problems in Set II (pages 1.18-19) in the Supplement.

vi) Power Series. The main concepts and results here are:

a) The radius of convergence. Every power series has a radius of convergence, i.e. there is always a number R so that $\sum a_n(x - x_0)^n$ converges absolutely for $|x - x_0| < R$ and diverges for $|x - x_0| > R$. We did not prove that. You need to show that if R is the least upper bound of the set of r's such that $\sum a_n(x - x_0)^n$ converges for $|x - x_0| \leq r$, then R will be the radius of convergence. That is pretty easy, but we did not have time for it. If the set of r's has no upper bound, then you need to show that the series is absolutely convergent for all x. In that case we say $R = \infty$. If you can use the ratio test on the power series, the radius of convergence is easy to find.

b) Given a function defined by a power series

$$f(x) = \sum_{n=0}^{\infty} a_n (x - x_0)^n, \quad \text{then } a_n = \frac{f^{(n)}(x_0)}{n!},$$

where $f^{(n)}(x)$ is the n-th derivative of $f(x)$. That is Theorem 3.2 in the Supplement.

The basic results on power series are concisely summarized in Section 5.1 of Boyce and DiPrima.

IV First Order Systems of Equations.

There is quite a bit of stuff in Boyce and DiPrima's Chapter 7, but it parallels Chapter 3. For us the main points are:

i) Any single equation $y^{(n)} + a_{n-1}(t)y^{(n-1)} + \cdots + a_1(t)y' + a_0(t)y = g(t)$ can be written as a first order system of n equations by simply defining $x_1 = y$, $x_2 = y', \ldots, x_n = y^{(n-1)}$. Then the system looks like $x_1' = x_2, x_2' = x_3, \ldots, x_{n-1}' = x_n, x_n' = -a_{n-1}(t)x_n - \cdots - a_1(t)x_2 - a_0(t)x_1 - g(t)$. If the orginal equation was inhomogeneous ($g \neq 0$), the system is inhomogeneous. If the original equation was homogeneous ($g = 0$), the system will be homogeneous. It is also sometimes useful to go the other way, and turn a system of n equations into a single equation of order n.

ii) Basic existence and uniqueness theory – and the Wronskian (Sections 7.1 and 7.4). This is like Sections 3.2 and 3.3. The existence and uniqueness theorem for first order systems takes the form: if $A(t)$ is an $n \times n$ matrix with entries that are continuous functions of t on the interval I and $\underline{g}(t)$ is an n-component vector of continuous functions on I, then, given t_0 in I, for every n-component vector $\underline{x}_0$, there is a unique vector-valued function $\underline{x}(t)$ such that $\underline{x}' = A(t)\underline{x} + \underline{g}(t)$ and $\underline{x}(t_0) = \underline{x}_0$ (Theorem 7.1.2). There is also a corresponding theorem for nonlinear first order systems (Theorem 7.1.1), but we never used that.

For a homogeneous system of n equations you need n solutions, $\underline{x}_1, \ldots, x_n$ for the Wronskian, and it is defined by

$$W(t) = \det \begin{pmatrix} | & & | \\ \underline{x}_1(t) & \cdots & \underline{x}_n(t) \\ | & & | \end{pmatrix}$$

With that definition the rest of the theory is exactly like Chapter 3. The same theorems from linear algebra tell you that the **vectors** $\underline{x}_1(t_0), \cdots, \underline{x}_n(t_0)$ are linearly independent and hence you can solve

$$c_1\underline{x}_1(t_0) + \cdots + c_n\underline{x}_n(t_0) = \underline{x}_0$$

for every n-vector $\underline{x}_0$ if and only if $W(t_0) \neq 0$. Likewise there is a version of Abel's theorem which tells you that $W(t)$ is either always zero on I or never zero. Hence you again reach the conclusion that $\underline{x}_1(t), \ldots, \underline{x}_n(t)$ will be a fundamental set of solutions on I if (and only if) its Wronskian is nonzero at one point in I.

iii) Constant Coefficient Systems. [Note that, since the coefficients are **constant**, they are continuous on the whole real line, and the interval I in the existence theorem is just $-\infty < t < \infty$.]

Here we studied the case of distinct eigenvalues: assume that the $n \times n$ matrix A has nonzero eigenvectors $\underline{v}_1, \ldots, \underline{v}_n$, $A\underline{v}_j = r_j\underline{v}_j$, $j = 1, \ldots, n$, and $r_j \neq r_k$ when j and k are different. In this case the $\underline{v}_j$'s are linearly independent (see "Matrices with distinct eigenvalues" on the class homepage or class notes) and $\underline{x}_j(t) = e^{r_j t}\underline{v}_j$, $j = 1, \ldots, n$, is a fundamental set of solutions to $\underline{x}' = A\underline{x}$. It is a **fundamental** set of solutions because its Wronskian is not zero when $t = 0$.

iv) The Matrix Exponential. For an $n \times n$ matrix A we defined

$$e^{At} = \sum_{n=0}^{\infty} \frac{t^n}{n!} A^n, \qquad \text{where } A^0 = I.$$

This is a nice (if somewhat mystifying) way of writing down the solution to $\underline{x}' = A\underline{x}$: we showed that the solution to $\underline{x}' = A\underline{x}$ with $\underline{x}(0) = \underline{x}_0$ is $\underline{x}(t) = e^{At}\underline{x}_0$. We have not yet shown that the power series in matrices **defining** e^{At} converges, but that is not difficult, and will be included if time permits.

When A has distinct eigenvalues as in iii), one can compute (and we did compute it!) that

$$e^{At} = \begin{pmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{pmatrix} \begin{pmatrix} e^{r_1 t} & & \\ O & \backslash & O \\ & & e^{r_n t} \end{pmatrix} \begin{pmatrix} | & & | \\ \underline{v}_1 & \cdots & \underline{v}_n \\ | & & | \end{pmatrix}^{-1}.$$

[The middle matrix there is supposed to have $e^{r_j t}$ running down the diagonal and zeroes everywhere else – but I couldn't make it look like that.] These results on the matrix exponential are intended to replace section 7.7 in Boyce and DiPrima. The matrix exponential is discussed in much more detail in the Supplement on pages 2.14 to 2.37. That was not used in the course, but it closely parallels what we did. In particular, you can find an alternative proof that eigenvectors belonging to distinct eigenvalues are linearly independent (Theorem 4.2 on page 2.34) in it.

In the case $n = 2$ when A has distinct **real** eigenvalues you can study the solutions of $\underline{x}' = A\underline{x}$ by graphing the parametric curves $(x_1(t), x_2(t))$ in the plane. The change of coordinates

$$\begin{pmatrix} y_1 \\ y_2 \end{pmatrix} = \begin{pmatrix} v_{11} & v_{12} \\ v_{21} & v_{22} \end{pmatrix}^{-1} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

makes this easy to do because in the y-coordinates the solution is just $(y_1(t), y_2(t)) = (e^{r_1 t} y_1(0), e^{r_2 t} y_2(0))$. That was done in class, Friday, Dec. 3.

v) The Complex Eigenvalue Case (Section 7.6). This will be covered Monday, Dec. 6. There is really just one observation: when the entries in A are real, complex eigenvalues have to occur in pairs. If $A\underline{v} = r\underline{v}$, then just taking the complex conjugate of every component in that equation gives you $A\overline{\underline{v}} = \overline{r}\overline{\underline{v}}$. So $\overline{r}$ is also an eigenvalue and $\overline{\underline{v}}$ is a nonzero eigenvector for it. This makes it possible to get fundamental sets of solutions with real entries. However, for small systems of equations, particularly 2×2 ones, it is really easier just to change the system to a single equation, solve that, and then go back to solutions of the system following i) above.

vi) Variation of Parameters for Systems (Section 7.9). This will be covered Wednesday, Dec. 8. Variation of parameters for first order systems of equations is actually **easier** than variation of parameters for second order equations. You just look for solutions to the inhomogeneous equation $\underline{x}' = A(t)\underline{x} + \underline{g}(t)$ in the form

$$\underline{x}(t) = c_1(t)\underline{x}_1(t) + \cdots + \underline{x}_n(t) = \begin{pmatrix} | & & | \\ \underline{x}_1(t) & \cdots & \underline{x}_n(t) \\ | & & | \end{pmatrix} \begin{pmatrix} c_1(t) \\ | \\ c_n(t) \end{pmatrix},$$

where $\underline{x}_1, \ldots, \underline{x}_n$ is a fundamental set of solutions for the homogeneous equation. Substituting that into the differential equation, you find that it solves the inhomogeneous equation when

$$\begin{pmatrix} c_1'(t) \\ | \\ c_n'(t) \end{pmatrix} = \begin{pmatrix} | & & | \\ \underline{x}_1(t) & \cdots & \underline{x}_n(t) \\ | & & | \end{pmatrix}^{-1} \begin{pmatrix} g_1(t) \\ | \\ g_n(t) \end{pmatrix}.$$

As in Section 3.9 this will lead you to integral formulas for solutions of the inhomogeneous equation. When you can do the integrals, the answers will be explicit formulas for the solutions. There are also undetermined coefficient methods for solving some special inhomogeneous equations. Problem 1 and 7 for section 7.9 listed in the syllabus were intended by Boyce and DiPrima to be done that way. However, I think it is really just as easy to solve them by rewriting them as second order equations for the unknown x_1, and solving that by the method of undetermined coefficients!

THAT'S ALL! BEST OF LUCK ON THE FINAL!