15. There have been quite a few questions in office hours about this one. Part of the difficulty is that the statement of the problem in Boyce & DiPrima disguises what is being asked for. Here is a more straightforward statement of that. You are given that

$$\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}' = A(t) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} \text{ and } \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}' = A(t) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}, \text{ where } A(t) = \begin{pmatrix} p_{11}(t) & p_{12}(t) \\ p_{21}(t) & p_{22}(t) \end{pmatrix}.$$

You are asked to show that the vector of functions defined by

$$\begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} c_1 x_1(t) + c_2 x_2(t) \\ c_1 y_1(t) + c_2 y_2(t) \end{pmatrix} \text{ satisfies } \begin{pmatrix} x \\ y \end{pmatrix}' = A(t) \begin{pmatrix} x \\ y \end{pmatrix}.$$

$c_1$ and $c_2$ are constants.

This is a very basic result. It says that linear combinations of solutions to a homogeneous $2 \times 2$ system are still solutions of that system. We use this for homogeneous systems of all sizes all the time. Still it is the sort of question that leaves you wondering where to begin. START WITH ONE SIDE OF THE EQUATION AND WORK UNTIL YOU GET TO THE OTHER SIDE. It is also a good idea to give reasons for what you do along the way. It looks like this:

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} c_1 x_1 + c_2 x_2 \\ c_1 y_1 + c_2 y_2 \end{pmatrix}' \text{ (substitution from what was given)}$$

$$= \begin{pmatrix} (c_1 x_1 + c_2 x_2)' \\ (c_1 y_1 + c_2 y_2)' \end{pmatrix} \text{ (definition of derivative of a vector function)}$$

$$= \begin{pmatrix} c_1 x_1' + c_2 x_2' \\ c_1 y_1' + c_2 y_2' \end{pmatrix} \text{ (the c's are constants and derivative of sum is sum of derivatives)}$$

$$= c_1 \begin{pmatrix} x_1' \\ y_1' \end{pmatrix} + c_2 \begin{pmatrix} x_2' \\ y_2' \end{pmatrix} \text{ (definitions of vector addition \& scalar multiplication)}$$

$$= c_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix}' + c_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix}' \text{ (definition of derivative of a vector function, again)}$$

$$= c_1 A(t) \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c_2 A(t) \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \text{ (substitution from what was given)}$$

$$= A(t)c_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + A(t)c_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \text{ (scalar mult. commutes with matrix mult.)}$$

$$= A(t) \left[ c_1 \begin{pmatrix} x_1 \\ y_1 \end{pmatrix} + c_2 \begin{pmatrix} x_2 \\ y_2 \end{pmatrix} \right] \text{ (matrix multiplication distributes over vector addition)}$$

$$= A(t) \begin{pmatrix} c_1x_1 + c_2x_2 \\ c_1y_1 + c_2y_2 \end{pmatrix} \text{ (vector addition \& scalar multiplication, again)}$$

$$= A(t) \begin{pmatrix} x \\ y \end{pmatrix} \text{ (substitution from what was given)}$$

OK, that was really exaggerated. I would never ask for that much detail on an exam. Working methodically from

$$\begin{pmatrix} x \\ y \end{pmatrix}' \text{ until you get } A(t) \begin{pmatrix} x \\ y \end{pmatrix},$$

mentioning when you are using what was given, is the point.