8. Find the general solution to $y'' + y = 3\sin 2t + t\cos 2t$.

Solution. There are at least two ways to do this. They both use the following principle: if you find a y_1 that solves

$$y'' + y = 3\sin 2t$$

and a y_2 that solves

$$y'' + y = t\cos t,$$

then $y_1 + y_2$ will solve the equation above. To find y_1 you can solve

$$w'' + w = 3e^{2it}$$

and take y_1 to be the imaginary part of w. That gives $y_1 = -\sin 2t$. Looking for $y_1 = A\sin 2t + B\sin 2t$ – as Boyce and DiPrima suggest – is really easier here, though neither method is difficult.

For y_2 you can solve

$$w'' + w = te^{2it},$$

and take y_2 to be the real part of w. To find w here try $w = (At + B)e^{2it}$. Then (see "Concise Version of Undetermined Coefficients" on the homepage or class notes) you need to equate coefficients in

$$(r^2 + pr + q)(At + B) + (2r + p)(At + B)' + (At + B)'' = t$$

in the case $p = 0$, $q = 1$ and $r = 2i$. That just gives the equation

$$(-3)(At + B) + 4iA = t$$

So $A = -1/3$ and $B = -4i/9$, and the real part of $(At+B)e^{2it}$ is $(-1/3)t\cos 2t + (4/9)\sin 2t$. Thus, adding up y_1, y_2 and the general solution to the homogeneous equation (which is $c_1\cos t + c_2\sin t$), we get

Answer:

$$y = -\sin 2t + (-1/3)t\cos 2t + (4/9)\sin 2t + c_1\cos t + c_2\sin t.$$

Note that the two "$\sin 2t$" terms can be combined to give $(-5/9)\sin 2t$.