26. Prove that if y_1 and y_2 have a common point of inflection t_0 in I, then they cannot be a fundamental set of solutions on I unless $p(t_0) = q(t_0) = 0$.

Solution. To simplify notation let's set $p(t_0) = p_0$ and $q(t_0) = q_0$. Then, since $y''(t_0) = 0$ when y has an inflection at t_0 (definition), the differential equation implies that

$$p_0 y_1'(t_0) + q_0 y_1(t_0) = 0 = p_0 y_2'(t_0) + q_0 y_2(t_0).$$

If $p_0 \neq 0$, this says $y_1'(t_0) = -(q_0/p_0)y_1(t_0)$ and $y_2'(t_0) = -(q_0/p_0)y_2(t_0)$. Substituting these relations in the formula for the Wronskian of y_1 and y_2 at t_0, you will see that the Wronskian is zero. If $q_0 \neq 0$, then $y_1(t_0) = -(p_0/q_0)y_1'(t_0)$ and $y_2(t_0) = -(p_0/q_0)y_2'(t_0)$. Substituting these relations in the formula for the Wronskian of y_1 and y_2 at t_0, you will see that the Wronskian is zero again.

So the Wronskian is zero at t_0 unless p_0 and q_0 are both zero. Thus y_1 and y_2 are linearly dependent on I unless p_0 and q_0 are both zero, and linearly dependent solutions can never be a fundamental set.