1.(8 pts.) Find the solution of $y' = 2y^2$ satisfing $y(1) = 1$.

Solution. The only constant solution is $y(x) = 0$ for all x which does not satisfy $y(1) = 1$. So we can find this solution by the usual method:

$$\int \frac{1}{y^2} dy = \int 2dx, \text{ so } \frac{-1}{y} = 2x + C.$$

Since $1 = y(1)$, this implies $-1 = 2 + C$, and $C = -3$. Solving for y gives the answer.

Ans. $y = \frac{1}{3-2x}$.

b) (2 pts.)What is the largest interval on the line where this solution is defined?

Solution. $(3-2x)^{-1}$ is defined for $x \neq 3/2$. The interval needed here is the largest interval containing $x = 1$ which does not contain $x = 3/2$.

Ans. $-\infty < x < 3/2$.

2. For both the equations below find the general solution AND the solution satisfying $y(0) = 0$ and $y'(0) = 1$.

a) (5 pts.) $y'' - 6y' + 10y = 0$

Solution. The characteristic equation here is $r^2 - 6r + 10 = 0$ which by the quadratic formula has roots

$$r_{\pm} = \frac{6 \pm \sqrt{36 - 40}}{2} = 3 \pm i$$

This makes the general solution

$$y = e^{3t}(c_1 \cos t + c_2 \sin t),$$

and the solution satisfying $y(0) = 0$ and $y'(0) = 1$ is $y = e^{3t} \sin t$.

b) (5 pts.) $y'' + 4y' + 4y = 0$

Solution. This time the characteristic equation is $r^2 + 4r + 4 = 0$ which has the repeated root $r = -2$. So the general solution is

$$y = e^{-2t}(c_1 + c_2 t),$$

and the solution satisfying $y(0) = 0$ and $y'(0) = 1$ is $y = te^{-2t}$.

3.(10 pts.) Suppose that the fish population at time t (in years) in Santa Monica Bay, $P(t)$ in the absence of any fishing satisfies the differential equation $\frac{dP}{dt} = (0.5)P$. Suppose that fish are caught continuously in the Bay at a constant rate of r fish per year. If there are 50,000 fish in the Bay now, find the largest value of r which will not lead to the extinction of the fish.

Solution. The main step here was to recognize that, it the fish are being caught continuously at rate r, then the rate of change of the fish population satisfies

$$\frac{dP}{dt} = (0.5)P - r. \text{ or } \frac{dP}{dt} - \frac{1}{2}P = -r$$

Then you could solve this by using the integrating factor $e^{-t/2}$. That gives

$$\frac{d}{dt}(e^{-t/2}P) = -re^{-t/2}, \text{ and}$$

$$P = 2r + Ce^{t/2} \text{ and, since } P(0) = 50,000,\ C = 50,000 - 2r.$$

So $P(t) = 2r + (50,000 - 2r)e^{t/2}$. If $50,000 - 2r > 0$, the fish population will continue to grow, but, if $50,000 - 2r < 0$, it will eventually become negative which means there are more fish. So the maximum fishing rate that does not lead to extinction of the fish is $r = 25,000$.

There are a couple remarks that could be made here. Just looking at the differential equation one sees that $P = 2r$ is the constant solution. So if $P = 50,000$ and $r = 25,000$, the fish population will stay at $50,000$. However, to see that the population will actually go to zero when r is greater than 25,000, you need to do more.

This answer depends on the population growing at a constant rate when there is no fishing and the fishing taking place at a constant rate. In a real population one can expect that the rate of growth will change with the season. One could also ask what would happen if certain parts of the year were closed to fishing. For instance if fishing were only allowed on New Year's Day, the fish population would grow to $50,000e^{1/2} = 82,436$ by December 31 and 32,436 fish could be caught (in the one fishing day) every year.

4.(10 pts.) This a problem about the equations

$$y'' + p(t)y' + q(t)y = f(t) \qquad (A)$$

and

$$y'' + p(t)y' + q(t)y = 0, \qquad (B)$$

where $f(t)$ is not zero. Assume that y_1 and y_2 are solutions of (A) and y_3 and y_4 are solutions of (B). For each linear combination of these functions below state whether it satisfies (A), (B) or neither. Justification of your answer is not required

Solution: This one befuddled quite a few of you. It is really very easy, and (trust me!) fundamentally important. The way to do it – until you understand the general picture – is just to substitute all functions into the equation. For instance for a) below you would have

$$(y_1 - y_2)'' + p(t)(y_1 - y_2)' + q(t)(y_1 - y_2) =$$

$$(y_1'' + p(t)y_1' + q(t)y_1) - (y_2'' + p(t)y_2' + q(t)y_2) = f(t) - f(t) = 0.$$

So $y_1 - y_2$ is a solution of (B). If you look at how that worked, you may see the general pattern already. The answers are:

a) $y_1 - y_2$ Ans. (B)

b) $y_1 + y_2$ Ans. Neither. This satisfies $y'' + p(t)y' + q(t)y = 2f(t)$.

c)$y_1 + y_3 + y_4$ Ans. (A)

d)$y_1/2 + y_2/2 + y_3/2 + y_4/2$ Ans. (A)

e)$25y_1 + 24y_2 + 23y_3 - 72y_4$ Ans. Neither. This satisfies $y'' + p(t)y' + q(t)y = 49f(t)$.

5. Suppose that y_1 and y_2 are solutions to

$$y'' + p(x)y' + q(x)y = 0$$

on an interval I where p and q are continuous. Assume that at the point x_0 in I we have $y_1(x_0) = 2$, $y_1'(x_0) = -1$, $y_2(x_0) = 1$ and $y_2'(x_0) = 1$. Prove the following statements. You will need to use theorems from Boyce and DiPrima or the lectures. State these theorems when you use them.

a) (5 pts.) y_1 and y_2 are linearly independent.

Solution. [Most people got this part.] The Wronskian of y_1 and y_2 at x_0 is

$$y_1(x_0)y_2'(x_0) - y_2(x_0)y_1'(x_0) = (2)(1) - (1)(-1) = 3 \neq 0.$$

Since the Wronskian at one point in the interval is not zero, y_1 and y_2 are linearly independent on the interval. This is the really easy theorem. It has nothing to do with Abel's Theorem. If y_1 is a constant multiple of y_2 or if y_2 is a constant multiple of y_1, it is very easy to see that their Wronskian will be zero everywhere. So if their Wronskian is nonzero anywhere one cannot be a multiple of the other, and they are linearly independent.

b) (5 pts.) Assume that x_1 is another point in I. Then, for every choice of the numbers a and b, there are constants c_1 and c_2 such that

$$c_1 y_1(x_1) + c_2 y_2(x_1) = a \text{ and } c_1 y_1'(x_1) + c_2 y_2'(x_1) = b.$$

Solution. [Very few people got this part.] In part a) we concluded that y_1 and y_2 were linearly independent by showing that their Wronskian was not zero at one point. So by Abel's Theorem their Wronskian cannot be zero at any point in I. In particular it is not zero at $x = x_1$. That means that the system of equations

$$c_1 y_1(x_1) + c_2 y_2(x_1) = a \text{ and } c_1 y_1'(x_1) + c_2 y_2'(x_1) = b$$

has a solution, namely

$$\begin{aligned} c_1 &= \frac{y_2'(x_1)a - y_2(x_1)b}{y_1(x_1)y_2'(x_1) - y_2(x_1)y_1'(x_1)} \\ c_2 &= \frac{-y_1'(x_1)a + y_1(x_1)b}{y_1(x_1)y_2'(x_1) - y_2(x_1)y_1'(x_1)}. \end{aligned}$$

Note that the Wronskian of y_1 and y_2 appears as the denominator in these formulas which is why we need to know that it is not zero. That was the real point of the problem. Boyce and DiPrima talk about "fundamental sets" without much explanation of why being a fundamental set at one point x_0 in I implies that they are a fundamental set at every x_1 in I. The reason is Abel's Theorem: once you know that the Wronskian of y_1 and y_2 is nonzero at x_0 it must be nonzero at every x_1 in I.

There were several possible variations on this argument. In particular, instead of writing out the formulas for c_1 and c_2, you could just point out that in class I showed that one can solve

$$c_1 y_1(x_1) + c_2 y_2(x_1) = a \text{ and } c_1 y_1'(x_1) + c_2 y_2'(x_1) = b$$

for all choices of a and b, if and only if the Wronskian of y_1 and y_2 at x_1 is not zero.