25.A body of (constant) mass m is projected vertically upward with an initial velocity v_0 in a medium which put a resisting force of $k|v|$ on the body. Here k is a constant. Assume that all of this happens close enough to the surface of the earth that the force of gravity can be assumed to be constant.

a) Find the maximum height x_m attained by the body and the time t_m that it reaches this height.

Solution. Since the body is going up for $0 \leq t \leq t_m$, we have $v \geq 0$ on this time interval and $|v| = v$. So the differential equation from Newton's Law is just

$$m\frac{dv}{dt} = -mg - kv \text{ or } \frac{dv}{dt} + \frac{k}{m}v = -g,$$

where g is the acceleration of gravity. The solution to that is

$$v(t) = -\frac{gm}{k} + Ce^{-kt/m} \text{ and, since } v(0) = v_0, \; C = v_0 + \frac{gm}{k}.$$

So

$$v(t) = -\frac{gm}{k} + (\frac{gm}{k} + v_0)e^{-kt/m}.$$

To find t_m we have to solve $0 = v(t_m)$ for t_m. That gives.

$$t_m = \frac{-m}{k}\ln(\frac{gm/k}{gm/k + v_0}) = \frac{m}{k}\ln(1 + \frac{kv_0}{mg}).$$

To find x_m use

$$x_m = \int_0^{t_m} v(t)dt = -\frac{gm}{k}t_m + (\frac{gm^2}{k^2} + \frac{mv_0}{k})(1 - e^{-kt_m/m}) = -\frac{gm^2}{k^2}\ln(1 + \frac{kv_0}{mg}) + \frac{mv_0}{k}.$$

[It takes a lot of algebra to get the final form for x_m.]

[The formulas in part b) of this problem are just the power series expansions of these answers.]