Comments on Homework 1:

Exercise 1.2: Most people had no difficulty proving the Contraction Mapping Theorem, but some points were lost for omitting details that I thought were important or for skipping parts of the problem.

Exercise 1.7: This one had a hidden difficulty. If you follow the hint, you come up against the problem of showing that

$$w(t) = \max\{u(t) - v(t), 0\}$$

has $\dot{w} \leq 0$ a.e. on the set where $u(t) - v(t) \leq 0$. That is easy enough in the interior of that set (!), but as a general closed set it can have boundary with positive measure. That leaves one with a question one would rather not think about – and no one did. However, the result needed here that $\partial_t w = 0$ a.e. on the set where $w = 0$ is true. For this result in all dimensions see Evans's Partial Differential Equations,Problem 17, pg. 291, where he gives you a nice hint on how to prove it.

Since in this excercise we are in one dimension, there is an easier way. Consider the set $E = \{t \in I : u(t) > v(t)\}$. This is open since u and v are continuous and hence consists of a countable union of disjoint open intervals. Let (a, b) be one of those intervals, and define $w(t) = u(t) - v(t)$ on $[a, b]$. If $a > t_0$, then $w(a) = 0$ by continuity, and, if $a = t_0$, the hypothesis $v(t_0) \geq u(t_0)$ and continuity imply that $w(a) = 0$ in that case, too. Now just apply Gronwall on $[a, b]$ to conclude that $w(t) \leq 0$ on $[a, b]$. This contradicts $(a, b) \subset E$, and we can conclude that E is empty, completing the proof.

For the case $v(t_0) > u(t_0)$ you can just use the preceding case as follows. Let $\tilde{u}$ and $\tilde{v}$ be the solutions to $\dot{w} = F(t, w(t))$ with data $\tilde{u}(t_0) = u(t_0)$ and $\tilde{v}(t_0) = v(t_0)$. Then by the preceding case we have

$$u(t) \leq \tilde{u}(t) \leq \tilde{v}(t) \leq v(t), \tag{1}$$

as long as $\tilde{u}$ and $\tilde{v}$ exist. However, (1) shows that $|\tilde{u}|$ and $|\tilde{v}|$ are bounded where they are defined by the maximum of $|u| + |v|$ on I, and hence cannot blow up. So we can assume $\tilde{u}$ and $\tilde{v}$ are defined on I. Since $\tilde{v}(t_0) > \tilde{u}(t_0)$, the uniqueness theorem says that $\tilde{v}(t) \neq \tilde{u}(t)$ on I. Thus $\tilde{v}(t) > \tilde{u}(t)$ on I, and by (1) the same inequality holds for $v(t)$ and $u(t)$, completing the proof.

1.8 Everyone followed the hint and realized that arguing by contradiction it sufficed to consider the case $u(t) > 0$ on (t_1, t_2) with $u(t_1) = u(t_2) = 0$ and $v(t) > 0$ on $[t_1, t_2]$. However, many people failed to notice that the inequalities $\dot{u}(t_1) \geq 0$ and $\dot{u}(t_2) \leq 0$ will not (directly) lead to the desired contradiction. The *strict* inequalities will, and $\dot{u}(t_1) = 0$ or $\dot{u}(t_2) = 0$ implies that $u = 0$ on I by the uniqueness theorem. So the strict inequalities hold by the assumption $u(t) > 0$ on (t_1, t_2).